Math 61: Introduction to Discrete Structures
Midterm #1

Instructor: Spencer Unger

October 27, 2014

Name: Solutions Version 1
ID # __________
Section __________
Good Luck! Be sure to justify your answers!
No calculators, books or notes are allowed.

Problem	Points	Score
1	20	
2	20	
3	10	
4	10	
5	20	
6	20	
Total	100	

1. (20 points) Show that for all $n \geq 1$, $8^n - 3^n$ is divisible by 5.

Base Case $n = 1$

$8^1 - 3^1 = 5$ which is divisible by 5

Induction step

Assume $8^n - 3^n$ is divisible by 5.

Look at $8^{n+1} - 3^{n+1}$

$$
\begin{aligned}
&= 8 \cdot 8^n - 3 \cdot 3^n \\
&= 5 \cdot 8^n + 3 \cdot 8^n - 3 \cdot 3^n \\
&= 5 \cdot 8^n + 3(8^n - 3^n)
\end{aligned}
$$

← Uses the induction hypothesis.

Both divisible by 5

2. (20 points)

(a) Throughout this problem use following sets:

$$
\begin{aligned}
A =& \{x \in \mathbb{R} \mid -3 \le \lfloor x \rfloor \le 17\} \\
B =& \{y \in \mathbb{Z} \mid -5 < y < 10\} \\
C =& \{z \in \mathbb{R} \mid z^2 \le 100\} \\
D =& \{w \in \mathbb{R} \mid w < -9\} \\
E =& \{n \in \mathbb{N} \mid n^2 + 1 \text{ is even }\}
\end{aligned}
$$

For each of the following statements determine whether it is true or false. Just write T or F for each.

i. $A \subseteq B$

F

ii. $C \cap D = \emptyset$

F

iii. $\{5\} \subseteq E \cap B$

T

iv. $10 \in C - D$

T

v. $17.5 \in A$

T

Parts (b) and (c) on the next page.

(b) Let $T = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid -10 \leq m \leq 10 \text{ and } -10 \leq n \leq 10\}$ and $S = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m^2 + n^2 \leq 100\}$. Is it true that $S = T$? Justify your answer.

$(10,10) \in T$

but $(10,10) \notin S$ since $10^2 + 10^2 \nleq 100$

There are other examples.

(c) Let X be a finite set. Give the definitions of both $\mathcal{P}(X)$ (the powerset of X) and $|X|$.

$\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$

= the set of all subsets of X.

$|X|$ = the # of elements of X.

3. (10 points) Let $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be given by $f(n, x) = nx$.

(a) Is f one-to-one? Justify your answer.

No $\quad f(1,0) = f(0,1) = 0$

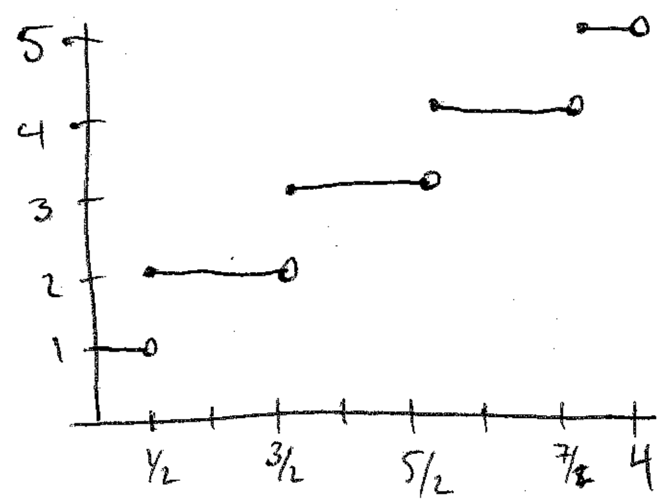
(b) Is f onto? Justify your answer.

Yes given $y \in \mathbb{R}$

$f(1,y) = 1 \cdot y = y$.

4. (10 points) Recall that for a real number x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Define a function $g : \mathbb{R} \to \mathbb{R}$ by the formula $g(x) = \lfloor x + \frac{3}{2} \rfloor$. From a homework problem we know that the relation R on $\mathbb{R}$ given by $(x, y) \in R$ exactly when $g(x) = g(y)$ is an equivalence relation. (You don't need to show this.) Answer the following questions:

(a) Graph the function g on the interval $[0, 4]$.



(b) The equivalence class of 3, [3] is an interval on the real line. Which interval is it?

$$[3] = \{x \mid (3,x) \in R\}$$
$$= [5/2, 7/2) \quad \text{from the picture.}$$

5. (20 points) Given two strings α and β we say that α is an *initial segment* of β if there is a string γ such that $\alpha\gamma = \beta$. Recall that $\alpha\gamma$ is the concatenation of α and γ. For example 011 is an initial segment of 01101, but 10 is not an initial segment of 11010.

   Let X be the set of binary strings of length at most 8. Define a relation R on X by (α, β) exactly when α is an initial segment of β. Answer the following questions about R. Be sure to justify your answers.

   (a) Is R reflexive? (Hint: There is a string of length 0, an empty string.)

   (b) Is R symmetric?

   Same as version 2

   (c) Is R transitive?

   (d) Is R antisymmetric?

6. (20 points) Work with a standard deck of cards. Recall that there are 52 cards and each card has a suit ($\diamondsuit, \heartsuit, \clubsuit, \spadesuit$) and a face value ($1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$). All combinations of suits and face values are possible.

   This problem will have you count the number of 5 card hands that have exactly 2 cards with face value 5 or exactly 3 cards with face value J. **For full credit please state any counting rules or principles that you use. You do not need to simplify your answers.**

   (a) Count the number of 5 card hands with exactly 2 cards of face value 5.

   (b) Count the number of 5 card hands with exactly three cards with face value J.

   Very similar to version 2.

   (c) Using your previous answer count the number of 5 card hands with exactly 2 cards of face value 5 or exactly 3 cards which have face value J. (Hint 1: Your count should include the hands that have exactly 2 cards with face value 5 *and* exactly 3 cards with face value J. Hint 2: Be careful!)

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Name: Solutions Version 2
ID # ____________________
Section ________
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Problem	Points	Score
1	20	
2	20	
3	10	
4	10	
5	20	
6	20	
Total	100	

1. (20 points) Show that for all $k \geq 1$, $7^k - 4^k$ is divisible by 3.

Base case: $k=1$

$7^1 - 4^1 = 3$ which is divisible by 3.

Induction Step:

Assume $7^k - 4^k$ is divisible by 3

Look at $7^{k+1} - 4^{k+1}$

$$= 7 \cdot 7^k - 4 \cdot 4^k$$

$$= 3 \cdot 7^k + \underbrace{4 \cdot 7^k - 4 \cdot 3^k}$$

Both divisible by 3

2. (20 points)

(a) Throughout this problem use following sets:

$$
\begin{aligned}
A =& \{x \in \mathbb{R} \mid -3 \leq \lfloor x \rfloor \leq 17\} \\
B =& \{y \in \mathbb{Z} \mid -5 < y < 10\} \\
C =& \{z \in \mathbb{R} \mid z^2 \leq 100\} \\
D =& \{w \in \mathbb{R} \mid w < -9\} \\
E =& \{n \in \mathbb{N} \mid n^2 + 1 \text{ is even }\}
\end{aligned}
$$

For each of the following statements determine whether it is true or false. Just write T or F for each.

i. $A \subseteq B$

F

ii. $C \cap D = \emptyset$

F

iii. $\{5\} \subseteq E \cap B$

T

iv. $10 \in C - D$

T

v. $17.5 \in A$

T

Parts (b) and (c) on the next page.

(b) Let $T = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid -10 \le m \le 10 \text{ and } -10 \le n \le 10\}$ and $S = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m^2 + n^2 \le 150\}$. Is it true that $S = T$? Justify your answer.

$S \neq T$

$(0,11) \in S$ since $0^2 + 11^2 = 121 \le 150$

but $(0,11) \notin T$ since 11 is not between -10 and 10.

There are other examples

(c) Let X be a finite set. Give the definitions of both $\mathcal{P}(X)$ (the powerset of X) and $|X|$.

$\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$

$|X|$ = the number of elements of X.

3. (10 points) Let $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be given by $f(n, x) = x^n$.

(a) Is f one-to-one? Justify your answer.

No

$f(2,1) = f(2,-1) = 1$.

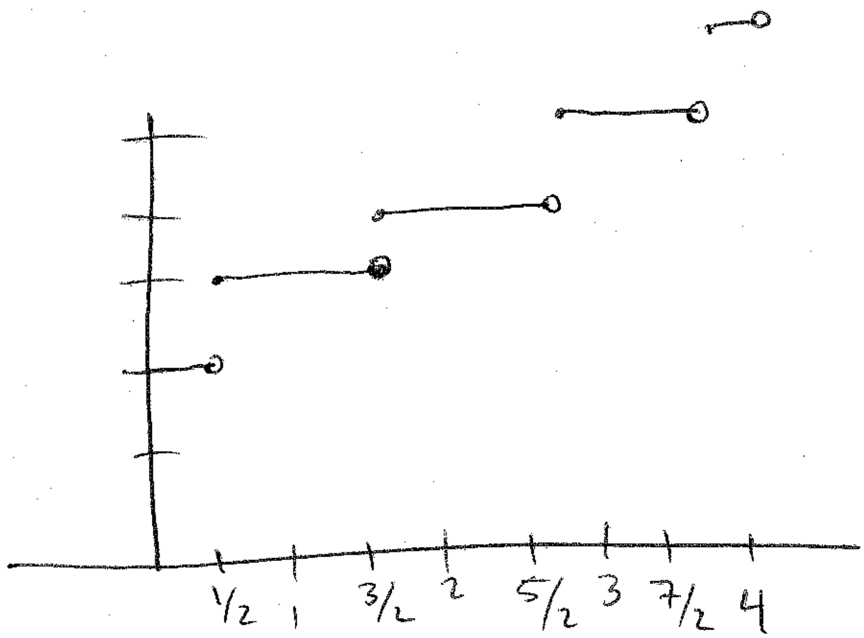
(b) Is f onto? Justify your answer.

Yes Let $z \in \mathbb{R}$

$f(1,z) = z^1 = z$.

4. (10 points) Recall that for a real number x, $\lceil x \rceil$ is the smallest integer greater than or equal to x. Define a function $g : \mathbb{R} \to \mathbb{R}$ by the formula $g(x) = \lceil x + \frac{3}{2} \rceil$. From a homework problem we know that the relation R on $\mathbb{R}$ given by $(x, y) \in R$ exactly when $g(x) = g(y)$ is an equivalence relation. (You don't need to show this.) Answer the following questions:

(a) Graph the function g on the interval $[0, 4]$.



(b) The equivalence class of 3, [3] is an interval on the real line. Which interval is it?

Using the problem

$[3] = [5/2, 7/2)$

5. (20 points) Given two strings α and β we say that α is an *initial segment* of β if there is a string γ such that $\alpha\gamma = \beta$. Recall that $\alpha\gamma$ is the concatenation of α and γ. For example 011 is an initial segment of 01101, but 10 is not an initial segment of 11010.

Let X be the set of binary strings of length at most 8. Define a relation R on X by (α, β) exactly when α is an initial segment of β. Answer the following questions about R. Be sure to justify your answers.

(a) Is R reflexive? (Hint: There is a string of length 0, an empty string.)

If γ is the empty string, then $\alpha \cdot \gamma = \alpha$ for any α so R is reflexive

(b) Is R symmetric?

No $(0, 01) \in R$ but $(01, 0) \notin R$

(c) Is R transitive?

Yes if $\alpha\gamma = \beta$ and $\beta\gamma' = \delta$, then $\alpha\gamma\gamma' = \delta$ so R is transitive.

(d) Is R antisymmetric?

Yes if $(\alpha, \beta) \in R$ and $(\beta, \alpha) \in R$ then α and β have the same length and are substrings of each other. Hence $\alpha = \beta$.

6. (20 points) Work with a standard deck of cards. Recall that there are 52 cards and each card has a suit ($\diamondsuit, \heartsuit, \clubsuit, \spadesuit$) and a face value ($1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$). All combinations of suits and face values are possible.

This problem will have you count the number of 5 card hands that have exactly 3 cards with face value 9 or exactly 2 cards with face value 10. **For full credit please state any counting rules or principles that you use. There is no need to simplify your answers.**

(a) Count the number of 5 card hands with exactly 3 cards of face value 9.

$$\binom{4}{3} \cdot \binom{48}{2}$$

choose 2 more cards that are not 9's

Choose 3 cards that are 9's

(b) Count the number of 5 card hands with exactly 2 cards with face value 10.

$$\binom{4}{2} \cdot \binom{48}{3}$$

choose 3 other cards

Choose 2 cards that are 10's

(c) Using your previous answer count the number of 5 card hands with exactly 3 cards of face value 9 or exactly 2 cards which have face value 10. (Hint 1: Your count should include the hands that have exactly 3 cards with face value 9 *and* exactly 2 cards with face value 10. Hint 2: Be careful!)

Use inclusion exclusion

$$\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{2}\binom{48}{3} - \overbrace{\binom{4}{2} \cdot \binom{4}{3}}$$