Math 61: Introduction to Discrete Structures
Midterm #2

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Name: Solutions
ID # ______
Section ______
Good Luck! Be sure to justify your answers!
No calculators, books or notes are allowed.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Version 1

1. Do the following:

   (a) (10 points) Solve the following recurrence relation:

$$a_n = 5a_{n-1} + 6a_{n-2}$$

   with initial conditions $a_0 = 9$ and $a_1 = 20$.

$$r^2 = 5r + 6$$

$$0 = r^2 - 5r - 6$$

$$0 = (r-6)(r+1)$$

$r = 6$ and $r = -1$

So $a_n = C \cdot 6^n + D(-1)^n$

$$9 = a_0 = C + D$$
$$20 = a_1 = 6C - D$$

$> \quad 29 = 7C$

$$C = \frac{29}{7}$$

So $D = 9 - C = \dfrac{63 - 29}{7} = \dfrac{34}{7}$

Finally $a_n = \left(\frac{29}{7}\right) 6^n + \left(\frac{34}{7}\right)(-1)^n$

Problem 1 is continued on the next page.

(b) (10 points) Let d_n be the number of strings of zeros and ones which do not contain 000 as a substring. Find a recurrence relation that d_n satisfies. Don't forget the initial conditions and be sure to explain which counting principles you used to obtain your answer.

There are three cases:

(1) The string starts with 1.

There are d_{n-1} strings for this case

(2) The string starts with 01.

There are d_{n-2} strings for this case

(3) The string starts with 001.

There are d_{n-3} strings for this case.

In each case we just fill in the number of remaining places with a string that avoids 000.

By the addition principle $d_n = d_{n-1} + d_{n-2} + d_{n-3}$

We need 3 initial conditions $d_1 = 2$, $d_2 = 4$, $d_3 = 7$.

2. (20 points) Do the following:

   (a) Imagine a trial in which you flip a coin 17 times in a row. Order matters in the outcome of the trial.

      i. Count the number of outcomes which have an even number of heads.

$$\sum_{k=0}^{8} \binom{17}{2k}$$

      ii. Count the number of outcomes which have an odd number of heads.

$$\sum_{k=0}^{8} \binom{17}{2k+1}$$

      iii. Find a simple formula for the sum of your answers from (a) and (b). Justify your answer.

The sum is the total # of trials which is 2^{17}.

Alternatively, the sum is the one that appears in the binomial theorem with $x=y=1$. The other side is $(x+y)^{17}$ which is 2^{17}.

Problem 2 is continued on the next page.

(b) Show that if you choose 13 numbers between 1 and 20 there must be two which differ by 5.

Let $a_1, \ldots a_{13}$ be 13 numbers between 1 and 20.

$a_1+5, a_2+5, \ldots a_{13}+5$ are all between 1 and 25.

So the combined lists give ~~25~~ 26 numbers between 1 and 25. So by the pigeon hole principle we must have $a_i = a_j + 5$ for some i, j.

So a_i and a_j differ by 5.

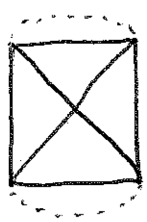
3. (20 points) Do the following:

(a) Consider the graph K_4. Answer the following questions:

i. What is the largest number of edges you can remove and still have a Hamilton cycle? Draw such a graph.

2 edges.

ii. What is the smallest number of edges you need to add (multiple edges allowed) to obtain a graph with an Euler cycle? Draw such a graph.

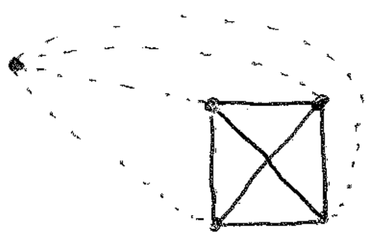


Added edges are dashed

2 edges.

iii. What are the smallest numbers of edges and vertices you need to add to create a simple graph (no multiple edges allowed) with an Euler cycle? Draw such a graph.

1 vertex and 4 edges



Added edges are dashed.

(b) A path is simple if it does not visit the same vertex twice. Count the number of simple paths in K_n of length 50.

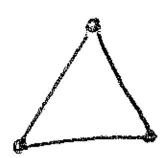
$$\frac{n!}{(n-51)!}$$

Since all edges are present in K_n, simple paths are exactly given by 50-arrangements from n.

4. (20 points) For the purpose of this question a graph is a simple graph, that is *no* self-loops and *no* multiple edges. A graph G is n-connected if it remains connected after removing any $n-1$ edges. Do the following:

(a) Draw a graph which is 1-connected, but not 2-connected.

(b) Draw a graph which is 2-connected, but not 3-connected



(c) Prove that if $n < k$ and a graph G is k-connected, then G is n-connected.

Let $n < k$ and G be k-connected.

After removing any $n-1$ edges the graph is still connected because $n-1 < k-1$ and G is k-connected.

So G is n-connected.

5. (20 points) Consider the graph $G = (V, E)$ where $V = \mathbb{R} \times \mathbb{R}$ and $\{x, y\} \in E$ if and only if the distance from x to y is 1. Answer the following questions about G.

(a) Answer true or false for each of the following:

i. $\{(0,0), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\} \in E$ T

ii. $\{(1,0), (0,1)\} \in E$ F

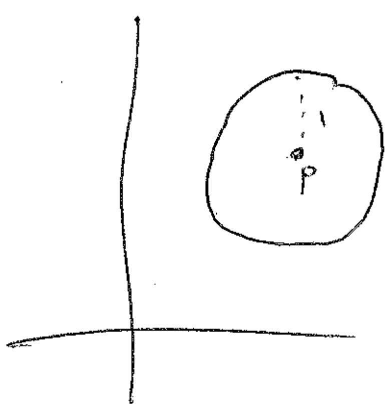
iii. There is a path of length 7 from $(0,0)$ to $(5,5)$. F

iv. There is a path of length 8 from $(0,0)$ to $(5,5)$. T

v. For any vertices p_1, p_2 and p_3, if $\{p_1, p_2\} \in E$ and $\{p_2, p_3\} \in E$, then $\{p_1, p_3\} \in E$. F

(b) Given a point $p \in V$, describe the set $\{x \in V \mid \{x, p\} \in E\}$. Drawing a picture is fine.

The circle of radius 1 centered at p.



(c) Prove or disprove: G is connected. This is true.

Let $p, q \in V$. Let the distance from p to q be x. Draw $\lfloor x \rfloor - 1$ edges from p in the direction of q. Let p' be the end of this path. The distance from p' to q is less than 2. So the circles of radius 1 centered at p' and q intersect. This completes the path.