Version 1

Math 61: Introduction to Discrete Structures
Midterm #1

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Name: Solutions
ID # ____________
Section ________
Good Luck! Be sure to justify your answers!
No calculators, books or notes are allowed.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (20 points) Let q be a fixed real number. Show by induction that the following formula holds for all $k \in \mathbb{N}$,

$$\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}$$

Base case $k=0$:

LHS $\sum_{i=0}^{0} q^i = q^0 = 1$

RHS $\frac{q^{0+1}-1}{q-1} = 1$

Induction Step:

Assume $\sum_{i=0}^{k} q^i = \frac{q^{k+1}-1}{q-1}$

$$\sum_{i=0}^{k+1} q^i = q^{k+1} + \sum_{i=0}^{k} q^i$$

Induction hypothesis.

$$= q^{k+1} + \frac{q^{k+1}-1}{q-1}$$

$$= \frac{q^{k+2}-q^{k+1}}{q-1} + \frac{q^{k+1}-1}{q-1}$$

$$= \frac{q^{k+2}-1}{q-1}$$

2. (20 points)

(a) Throughout this problem use following sets:

$$A = \{x \in \mathbb{Z} \mid -3 \le x < 17\}$$
$$B = \{y \in \mathbb{Z} \mid -5 < y < 3\}$$
$$C = \{z \in \mathbb{R} \mid z^2 \le 100\}$$
$$D = \{w \in \mathbb{R} \mid w < -3\}$$
$$E = \{n \in \mathbb{N} \mid n \text{ is even }\}$$

For each of the following statements determine whether it is true or false. Just write T or F for each.

i. $A \subseteq B$

F

ii. $C \cap D = \emptyset$

F

iii. $\{4\} \subseteq E \cap B$

F

iv. $10 \in C - D$

T

v. $E \cup B \subseteq \mathbb{Z}$

T

Parts (b) and (c) on the next page.

(b) Let $T = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m = n^2\}$ and $S = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m^2 = n\}$. Prove or disprove $S = T$.

$S \neq T$ since $(4,2) \in T$ b/c $4 = 2^2$

but $(4,2) \notin S$ b/c $4^2 \neq 2$.

(c) Recall that $\mathcal{P}(X)$ is the powerset of X. Determine whether the following statement is true or false and justify your answer: For all sets X and Y,

$$\mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y)$$

If $Z \in \mathcal{P}(X) \cup \mathcal{P}(Y)$ then either $Z \subseteq X$ or $Z \subseteq Y$

Case 1: $Z \subseteq X$

This means $Z \subseteq X \cup Y$ so $Z \in \mathcal{P}(X \cup Y)$

Case 2: $Z \subseteq Y$

This means $Z \subseteq X \cup Y$ so $Z \in \mathcal{P}(X \cup Y)$

In both cases we concluded $Z \in \mathcal{P}(X \cup Y)$

So the statement is true.

3. (20 points) Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = 2x - 3$ and $g : \mathbb{Z} \to \mathbb{N}$ be given by $g(z) = |z| + 4$. Be sure to justify your answers.

(a) What are the domain and codomain of $g \circ f$?

Domain $\mathbb{Z}$

Codomain $\mathbb{N}$

(b) Find a formula for $g \circ f$.

$$g \circ f(x) = g(f(x)) = g(2x-3) = |2x-3| + 4$$

(c) Is $g \circ f$ injective?

No

$$g \circ f(1) = |2(1) - 3| + 4 = 5$$

and $g \circ f(2) = |2(2) - 3| + 4 = 5$.

(d) Is $g \circ f$ surjective?

No 1 is not a member of the range

Since $|2x-3| \geq 0$ so $|2x-3| + 4 \geq 4$

The definition of $\lfloor x \rfloor$ should end with "... less than or equal to x."
Both versions are acceptable in your answer

4. (20 points) Recall that for a real number x, $\lfloor x \rfloor$ is the greatest integer less than x. Define $R = \{(r, s) \in \mathbb{R} \times \mathbb{R} \mid \lfloor r \rfloor = \lfloor s \rfloor\}$.

(a) Show that R is an equivalence relation.

For any x, $\lfloor x \rfloor = \lfloor x \rfloor$. So R is reflexive

For any x, y $\lfloor x \rfloor = \lfloor y \rfloor \longrightarrow \lfloor y \rfloor = \lfloor x \rfloor$ so R is symmetric.

For any x, y, z $\lfloor x \rfloor = \lfloor y \rfloor$ and $\lfloor y \rfloor = \lfloor z \rfloor$

$\longrightarrow \lfloor x \rfloor = \lfloor z \rfloor$

So R is transitive.

It follows that R is an equivalence relation.

(b) What is the equivalence class of 5.5? Write your answer in proper set notation.

If $\lfloor x \rfloor$ is defined with "... less than or equal to x."

Then $[5.5] = \{y \mid \lfloor 5.5 \rfloor = \lfloor y \rfloor\} = \{y \mid 5 = \lfloor y \rfloor\} = [5,6)$

If $\lfloor x \rfloor$ is defined with "... less than x."

then $[5.5] = \{y \mid \lfloor 5.5 \rfloor = \lfloor y \rfloor\} = \{y \mid 5 = \lfloor y \rfloor\} = (5,6]$

5. (20 points) A company issues ID numbers to each of its employees. The employees are broken up into 3 divisions, Divisions A, B and C. Each employee ID begins with the division letter that the employee belongs to. Within the divisions the IDs are arranged as follows:

- Division A is the executive division which only has 5 members. The ID numbers assigned are $A1$, $A2$, $A3$, $A4$ and $A5$.
- Division B is a division of poets whose ID numbers are the letter B followed by an arrangement of the 7 letters J, U, S, T, I, C and E.
- Division C is a division of computer programmers whose employee IDs are the letter C followed by a string of 0's and 1's of length 8.

If each employee gets a unique ID number and all the ID numbers are used, then how many employees does the company have? In your answer please explain which counting principles you used and how you used them.

The addition principle gives

\# of employees = \# of employees in Division A
+ \# of employees in Division B
+ \# of employees in Division C

\# in A ~~is~~ 5

\# in B is $7!$ using the multiplication principle

\# in C is 2^8 using the multiplication principle

So \# of employees is $5 + 7! + 2^8$.