Math 61 Final Exam
Winter quarter 2014

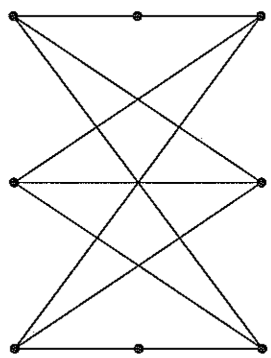
Instructor: Spencer Unger

March 20, 2014

Name: Solutions
ID # ____________________
Section ________
Good Luck! Be sure to justify your answers!
No calculators, books or notes are allowed.

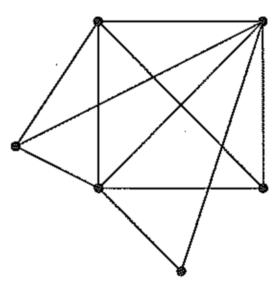
Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total	200	

1. (20 points) For parts (a) and (b), determine whether each of the following graphs is planar. If it is planar, show it by redrawing the graph. If it is not planar then explain why not using facts from lecture. Don't forget part (c).

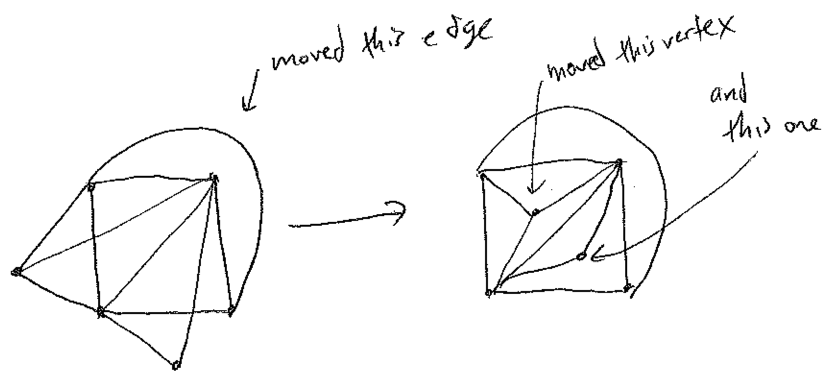


(a)

This graph is homeomorphic to $K_{3,3}$, therefore not planar.



(b)



This gives a planar representation.

(c) If either of the graphs from parts (a) or (b) was planar, then verify Euler's formula for that graph.

(b) was planar.

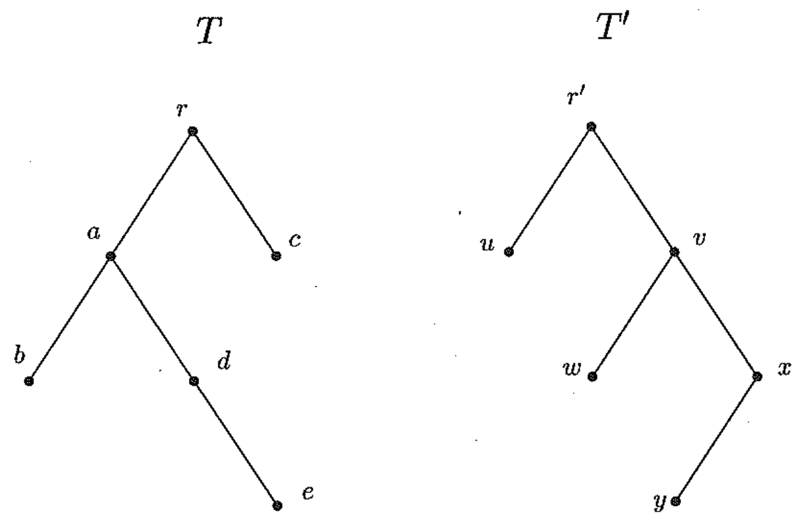
faces: from picture above, 7 (including unbounded face)

6 vertices

$\#\text{edges} = \dfrac{3+4+5+5+2+3}{2} = 11$

$\boxed{6-11+7=2}$

2. (20 points) Let $T = (V, E)$ be a tree with root r and $T' = (V', E')$ be a tree with root r'. A rooted-tree-isomorphism f from T to T' is a graph isomorphism from T to T' with the extra property that $f(r) = r'$. The following questions concern the rooted trees T with root r and T' with root r' pictured below.



(a) Find a rooted-tree-isomorphism from T to T'.

$f:$
$r \mapsto r'$ (roots)
$a \mapsto v$ (a is the child of r with two children)
$b \mapsto w$ (b is the child of a with no children)
$c \mapsto u$ (c is the child of r with no children)
$d \mapsto x$
$e \mapsto y$

(b) Find a graph isomorphism which is not a rooted-tree isomorphism.

We want $f(r) \neq r'$.

$r \mapsto x$ (this is the only possible choice, since r' and x are the only degree 2 vertizes in T')

$f:$
$a \mapsto v$
$b \mapsto w$
$c \mapsto y$
$d \mapsto r'$
$e \mapsto u$

3. (20 points) Assume the sequence of numbers a_n for $n \in \mathbb{N}$ satisfies the recurrence relation $a_n = a_{n-1} + 2n - 1$ with $a_0 = 0$. Show by induction that for all $n \in \mathbb{N}$, $a_n = n^2$.

Base Case:

$a_0 = 0 = 0^2$.

Induction step:

Let $N \in \mathbb{N}$ arbitrary.

Suppose $a_N = N^2$.

Then $a_{N+1} = a_N + 2(N+1) - 1$

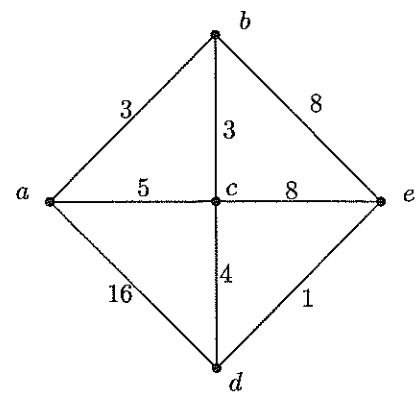
$= N^2 + 2N + 1$ (by induction hypothesis)

$= (N+1)^2$.

Therefore

$a_n = a_{n-1} + 2n - 1$ for all $n \in \mathbb{N}$.

4. (20 points) This question concerns the following weighted graph.



(a) In what order does the shortest path algorithm visit the vertices of the following graph while finding the shortest path from a to e?

Starting point:

1. a initializes the algorithm.

0 ⓐ

b: 3

c: 5

d: 16

e: ∞

2. b has the smallest label so far.

3 ⓑ

c: 5

d: 16

e: 11 (< ∞)

3. Now c.

5 ⓒ

d: 9 = 5+4 (< 16)

e: 11 (< 5+8 = 13)

4. Lastly, d.

9 ⓓ

e: 10 (< 11)

$\boxed{a\,b\,c\,d\,e}$

(b) Use Prim's algorithm starting at c to find a minimum spanning tree. Indicate the order in which you added the edges to the tree.

Start with c.

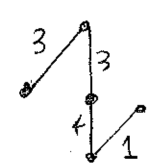
Vertices in Tree so far:

1. {b, c} is the ~~step~~ smallest weight edge to c. — b, c

2. {a, b} is the smallest weight edge between b, c and the other vertices — a, b, c

3. ~~{a}~~ {c, d} — a, b, c, d

4. {d, e} — a, b, c, d, e



5. . (20 points) Let $n \in \mathbb{N}$ and let A and B be sets of size n.

(a) Show that if $f : A \to B$ is one-to-one, then f is onto.

Suppose for a contradiction that f is not onto.

Let the pigeonholes be labeled by the range of f, and the pigeons be the elements of A.

Say a pigeon a is in pigeonhole b if $f(a)=b$.

By our assumption that f is not onto, there are only at most $n-1$ pigeonholes. By the pigeonhole principle, there must be $a \neq a'$ in A such that $f(a) = f(a')$, contradicting 1-to-1.

(b) Show that if $g : A \to B$ is onto, then g is one-to-one.

Suppose for a contradiction that g is not one-to-one, so there is a $b \in B$ with more than one $a \in A$ such that $g(a) = b$.

If we delete all such a's from the ~~range~~ domain of g, and b from its codomain, we are left with a function $\hat{g}$ with domain of size $< n-1$ and codomain of size $n-1$. Furthermore, $\hat{g}$ is onto, so its range has size $n-1$, which is larger than its domain, a contradiction.

6. (20 points) Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ which takes a point in the plane and rotates it $45°$ about the origin clockwise. Consider the graph $G = (V, E)$ where $V = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$ and $\{(x, y), (z, w)\} \in E$ exactly when either $F(x, y) = (z, w)$ or $F(z, w) = (x, y)$. Do the following:

(a) Determine whether each of the statements is true or false. Just write T or F for each.

i. $\{(1, 0), (\sqrt{2}, \sqrt{2})\} \in E$. F: $(\sqrt{2}, \sqrt{2}) \notin V$, since $\sqrt{2}^2 + \sqrt{2}^2 \neq 1$.

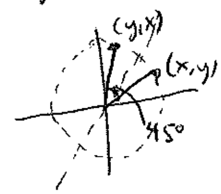
ii. $\{(0, 1), (-1, 0)\} \in E$. F: $90°$ rotation takes one to the other

iii. $\{(1, 1), (2, 0)\} \in E$. F: $(2,0) \notin V$, since $2^2 + 0^2 \neq 1$. Also, $(1,1) \notin V$.

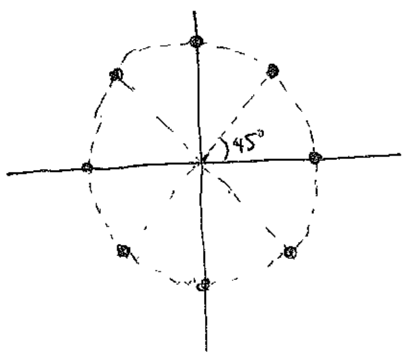
iv. G has a cycle of length 4. F: This would mean there is $(x,y) \in V$ which returns to itself after 4 $45°$ rotations

v. There are real numbers x and y such that $\{(x, y), (y, x)\} \in E$. T

(x,y) is the point (y,x) reflected about the line $y=x$.



(b) Exhibit a cycle of length 8 in G.

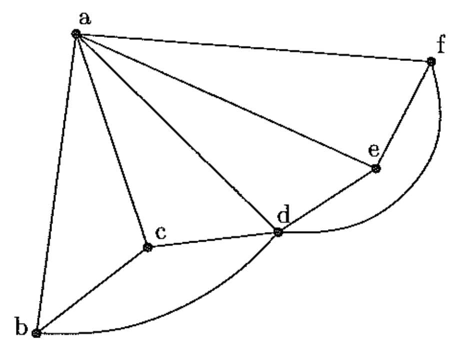


Consecutive vertices on the circle are adjacent in G.

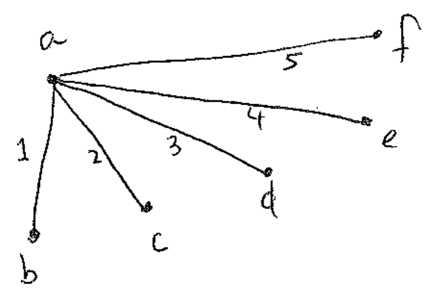
(c) Show that G is not connected.

The vertices in the cycle from part (b), for example, have no edges to other vertices. So there is no path from a vertex in the cycle to one outside.

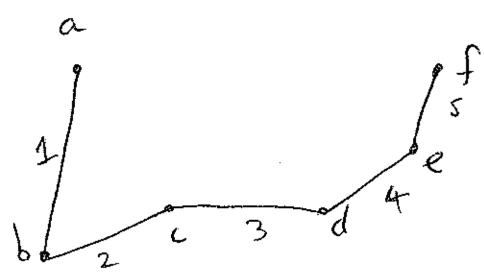
7. (20 points) Consider the following graph G. Use the alphabetical order on the vertices in the following questions.



(a) Draw the spanning tree obtained by breadth first search. Number the edges of the spanning tree in the order in which they were added to the tree.



(b) Draw the spanning tree obtained by depth first search. Number the edges of the spanning tree in the order in which they were added to the tree.



8. (20 points) Consider the following 5×5 adjacency matrix.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

As usual label the vertices 1 through 5. How many paths are there of length 2 from vertex 2 to vertex 3?

Vertex 2 is connected to vertices

1, 3, 4, 5.

Vertex 3 is connected to vertices

1, 2, 4 .

A path of length 2 from vertex 2 to vertex 3 can go through vertex 1, or through vertex 4.

So there are $\boxed{2}$ such paths.

9. (20 points) Define a binary relation R on $\mathbb{Q}$ by $(a, b) \in R$ exactly when $a \cdot b \geq 0$. For each of the following statements determine whether it is true or false and prove or disprove it accordingly.

(a) R is reflexive True.

Let $a \in \mathbb{Q}$ be arbitrary.

Then $a^2 \geq 0$, so $(a,a) \in R$.

Therefore R is reflexive.

(b) R is symmetric True.

Suppose $(a,b) \in R$.

Then $a \cdot b \geq 0$.

Since $a \cdot b = b \cdot a$, $b \cdot a \geq 0$.

Therefore, $(b,a) \in R$ so R is reflexive.

(c) R is transitive. False.

As a counterexample, $(-1, 0) \in R$ and $(0, 1) \in R$, but $(-1, 1) \notin R$.

10. (20 points) Consider the standard 26 letter English alphabet. 'Words' below refers to any arrangement of letters from the English alphabet. Answer the following questions.

(a) How many 3 letter words are there?

There are 26 choices for each letter.

$\boxed{26^3}$.

(b) How many 5 letter words begin with a vowel? (A vowel is one of A, E, I, O or U.)

There are 5 choices for the first letter, 26 choices for the others.

$\boxed{5 \cdot 26^4}$

(c) How many 6 letter words have at most 2 consonants? (A consonant is a letter that is not a vowel.)

Three cases:

Zero consonants: 5 choices for each letter. 5^6

One consonant: 6 choices for which letter is the consonant, 21 choices for that letter. 5 choices for the others. $6 \cdot 21 \cdot 5^5$

Two consonants: $\binom{6}{2}$ choices for which letter is the consonant. 21 choices for those letters, 5 for others. $\binom{6}{2} \cdot 21^2 \cdot 5^4$

Using the addition principle,

$\boxed{5^6 + 6 \cdot 21 \cdot 5^5 + \binom{6}{2} \cdot 21^2 \cdot 5^4}$

(d) How many 4 letter words have 4 distinct letters?

This is $\boxed{P\binom{26}{4} = 26 \cdot 25 \cdot 24 \cdot 23}$