B (green)

Discrete Structures
Math 61, Winter 2015 — Schaeffer
Midterm Exam 2

Name and Bruin ID:

In LARGE CAPITALS, the first 3 letters of your last/family name:

Circle your TA. If you do not know your TA's name, you must speak with Professor Schaeffer when you hand in your exam so he can look it up.

Zhu (A,B) Rosenbaum (C,D) Zhang (E,F)

Instructions: Complete all problems. Notes and electronics are not permitted. Good luck!

Problem	Notes	Grade
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
Total		

NOTE: Throughout this exam, *graph* means simple, undirected graph (no parallel edges or loops). Unless otherwise specified, you do not have to simplify your numerical answers.

1. *In a standard deck of cards there are* 4 *suits (club, heart, diamond, spade) and* 13 *ranks (A (ace), 2–10, J (jack), Q (queen), K (king)). There are* $4 \cdot 13 = 52$ *cards total.*

   A poker hand is a 5-combination taken from a standard deck of cards.

   a. In how many poker hands are there *exactly* two cards that are hearts?

   $$\binom{13}{2}\binom{39}{3}$$

   b. In how many poker hands are there *at least* two cards that are hearts?

   $$\binom{13}{2}\binom{39}{3} + \binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}\binom{39}{0}$$

   c. How many poker hands are there with at least one card of every suit?

   $$4 \cdot \binom{13}{2}\binom{13}{1}^3$$

2. Let $S = \{1, \ldots, 20\}$ and consider the following 9 quantities:

   - $A =$ the number of subsets U of S such that $|U| = 0$
   - $B =$ the number of subsets U of S such that $|U| = 5$.
   - $C =$ the number of subsets U of S such that $|U| = 5$ disjoint with $\{1, \ldots, 16\}$.
   - $D =$ the number of subsets U of S such that $|U| = 5$ disjoint with $\{1, \ldots, 5\}$.
   - $E =$ the number of subsets U of S containing only odd numbers.
   - $F =$ the number of subsets U of S containing only even numbers
   - $G =$ the number of subsets U of S that contain the number 5.
   - $H =$ the number of subsets U of S that do not contain the number 5.
   - $I =$ the number of subsets U of S such that $|U| > 0$

   Fill in the nine blanks below with $A, \ldots, I$ so that each entry is $\leq$ the next one.

   $C \leq A \leq F \leq E \leq D \leq B \leq G \leq H \leq I$

3. a. Consider the formula

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}$$

The left-hand side counts the number of strings over the alphabet $\{0, 1, 2\}$. What subset of those strings does $\binom{n}{k} 2^{n-k}$ on the right-hand side count?
There are many possible answers.

those w/ exactly k 0's

b. Name a theorem we covered in class that you could use to immediately prove the formula from part (a.) without writing a whole combinatorial proof.

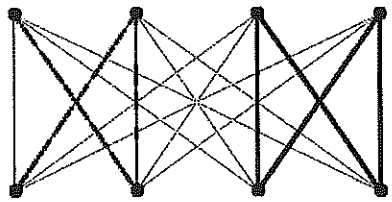
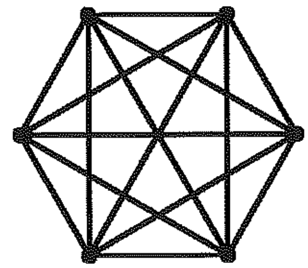
binomial theorem

4. "If A and B are finite sets and $|A| > |B|$, then any function $f : A \to B$ is not injective" is another way of stating one of the combinatorial principles we covered in class. *Which one?*

pigeonhole principle

5. Below, $G = (V, E)$ denotes a graph. *Circle all true statements below.*

a. If G is connected, then $|E| \geq |V|$.
b. There are no semi-Eulerian trees with ≥ 3 vertices.
c. There exists a Hamiltonian tree with ≥ 2 vertices.
d. Every tree with ≥ 2 vertices is bipartite.
e. If every vertex of G has degree k, then $|E| = k|V|$ edges.
f. If G is connected and bipartite, then $|E| \geq |V|$.
g. If G is bipartite and n is odd, then G is not Hamiltonian.
h. If $|V| \geq 2$ then there are $v, w \in V$ such that $v \neq w$ and $\deg(v) = \deg(w)$.

6. Above are the graphs $K_{4,4}$ and K_6 (sorry about the image quality—in $K_{4,4}$ there is an edge from every vertex in the 1st row to every vertex in the 2nd row, 16 edges total).

   a. Circle a vertex in $K_{4,4}$.
   How many Hamilton cycles are there in $K_{4,4}$ that begin at the circled vertex?

   $4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1$

   b. Circle a vertex in K_6.
   How many Hamilton cycles are there in K_6 that begin at the circled vertex?

   $5!$

7. Xavier and Yolanda are having fun solving second order linear recurrences with constant coefficients (SOLRwCCs, for short). They're currently trying to solve one where the characteristic equation has only one root, $r \neq 0$. Xavier's solution has the form

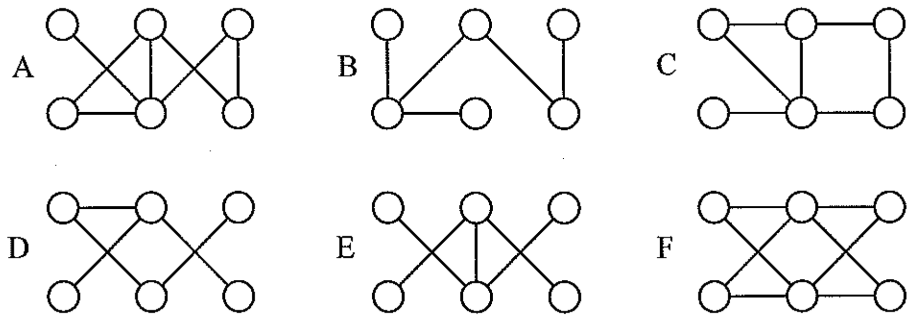
$$a_n = Pr^n + Qnr^n$$

for some constants P, Q, but(!!!) Yolanda's solution has the form

$$a_n = P'r^n + Q'nr^{n-1}$$

Before sending panicked emails, they stop for a moment to consider... Who is correct? *Circle one answer.*

   a. Xavier. This is the solution from the textbook; Professor Schaeffer is a liar!
   b. Yolanda. This is the solution from the notes; Professor Johnsonbaugh is a liar!
   c. They are both correct. I am at peace.



8. Consider the graphs in the figure above.

   a. Circle the letters corresponding to the graphs in the figure that are *Eulerian.*

   A B C D E F

   b. Circle the letters corresponding to the graphs in the figure that are *bipartite.*

   A B C D E F

9. In a.–d. below determine if the given two graphs (from the figure) are isomorphic.
   If they are isomorphic, write "ISOMORPHIC." If they are not isomorphic, write down one reason why they are not.

   a. A and C?

   iso.

   b. B and D?

   iso.

   c. B and E?

   not vertices of deg 3

   d. E and F?

   not vertices of deg 4

10. Define a sequence (a_n) as follows:

- $a_0 = 7$ and $a_1 = 5$,
- When $n \geq 2$, a_n is equal to the average of the last two terms in the sequence

(For example, $a_2 = \frac{7+5}{2} = 6$ and $a_3 = \frac{5+6}{2} = \frac{11}{2}$.)

Find a closed form for a_n. *Put a box around your final answer.*

$$a_n = \frac{1}{2} a_{n-1} + \frac{1}{2} a_{n-2}$$

so char eq. is $2x^2 - x - 1 = 0$

roots are

$$\frac{1 \pm \sqrt{1^2 - 4(2)(-1)}}{4} = \frac{1 \pm \sqrt{9}}{4}$$

$$= \frac{1 \pm 3}{4} = 1 \text{ or } -\frac{1}{2}$$

$$a_n = P(1)^n + Q\left(-\frac{1}{2}\right)^n$$

$a_0 = P + Q = 7$ so $P = 7 - Q$

$a_1 = P - \frac{1}{2}Q = 5$ $\quad 5 = (7-Q) - \frac{1}{2}Q$

$$\boxed{}$$

$2 = \frac{3}{2}Q$ $\quad Q = \frac{4}{3}$ $\quad P = \frac{21-4}{3} = \frac{17}{3}$.

Form Blue

(2)

X_1	C	
X_2	A	
X_3	G	✓
X_4	F	✓
X_5	D	
X_6	B	
X_7	H	
X_8	I	
X_9	E	

Circle equal pairs

to check: C E

2. A ≤ G

3. H ≤ I ≤ ...

4. ~~F ≤ G~~ . G ≤ F .

(5)

a

b

c

d

e

f

g

h

(8)

a A B C D E F

b A B C D E F

(9)

a. ✓

b. ×

c. ✓

d. X

(10) Roots: 1 $-\frac{1}{2}$

Constants: 5 4

4 3 3 2 2 1 3 6

Form Green

(2)

x_1	C
x_2	A
x_3	F
x_4	E
x_5	D
x_6	B
x_7	G
x_8	H
x_9	I

Circle equal pairs

(5)

a

b

c

d

e

f

g

h

(8)

a A B C D E F

b A B C D E F

(9)

a. iso.

b. iso.

c. different # of vertices deg 3

d. different # of vertices deg 4

(10) Roots: $1, -\frac{1}{2}$

Constants: $\frac{7}{3}, \frac{4}{3}$

Form PINK

(2)

X_1 D

X_2 A

X_3 G

X_4 F

X_5 E

X_6 B

X_7 H

X_8 I

X_9 C

Circle equal pairs

(5)

a

b

c

d

e

f

g

h

(8)

a A B C D E F

b A B C D E F

(9)

a. Iso

b. B has no cycles

c. B has 2 vtx of deg 3

d. Iso

(10) Roots: $1, -\frac{1}{2}$

Constants: $\frac{10}{3}, -\frac{4}{3}$

Form Yellow

(2)

x_1	D
x_2	A
x_3	F
x_4	I
x_5	C
x_6	B
x_7	G
x_8	H
x_9	E

Circle equal pairs

(5)

a	✓	O	a
b	X	—	b
c	✓	O	c
d	✓	O	d
e	✓	O	e
f	X	—	f
g	✓	O	g
h	X	—	h

(8)

a A B C D E F

b A B C D E F

(9)

a. X

b. ✓

c. X

d. ✓

(10) Roots: $1, -\frac{1}{2}$

Constants: $\frac{11}{3}, -\frac{8}{3}\left(-\frac{1}{2}\right)^n$