61 Final Exam

Alvin Duong

TOTAL POINTS

94 / 120

QUESTION 1

1 True/False 8 / 10

- **- 2 pts** (1) Incorrect [Correct answer: False]
- **- 2 pts** (2) Incorrect [Correct answer: False]
- ✓ **- 2 pts (3) Incorrect [Correct answer: False]**
- **- 2 pts** (4) Incorrect [Correct answer: False]
- **- 2 pts** (5) Incorrect [Correct answer: True]
- **- 0 pts** Correct

QUESTION 2

2 Sum of power set cardinalities 10 / 10

- ✓ **- 0 pts Correct**
- **- 10 pts** not graded
- **- 3 pts** Not enough justification
- **- 1 pts** arithmetic error
- **- 7 pts** major incorrect reasoning

QUESTION 3

3 Sum of combinations 10 / 10

- ✓ **- 0 pts Correct**
- **- 10 pts** Not graded
- **- 2 pts** missing minor details in reasoning
- **- 5 pts** Incomplete
- **- 7 pts** Major gap missing

QUESTION 4

4 Straight/flush 10 / 10

- ✓ **+ 10 pts Fully/mostly correct**
- **+ 0 pts** Skipped
- **+ 3 pts** counted straights
- **+ 1 pts** [partial credit] Some progress counting straights
- **+ 3 pts** counted flushes
- **+ 1 pts** [partial credit] Some progress counting flushes. (e.g. forgot to include the suits in the count, or counted where order mattered)
- **+ 2 pts** Counted straight flushes
- **+ 2 pts** Used PIE correctly
- **- 1 pts** Small mistake counting straights
- **- 1 pts** Small mistake counting flushes
- **- 1 pts** Small mistake counting straight flushes
- **+ 1 pts** [partial credit] Some progress counting straight flushes.
- **- 2 pts** No work shown
- **+ 0 pts** Error when doing unnecessary calculation.

QUESTION 5

5 Integer solutions 10 / 10

- ✓ **- 0 pts Correct with valid work**
- **- 2 pts** Minor error (e.g. off by 1 error in the combinations)
- **- 4 pts** Correct calculations for correct cases, but incorrect combination of numbers
- **- 5 pts** Flipped the inequality in the second condition and solved the resulting problem correctly, but in a way that does not scale to the correct problem
- **- 6 pts** Incorrect specification of cases or calculations for said cases
- **- 8 pts** Major errors in setting up cases, counting numbers of solutions in cases, and/or combining the results
- **- 8 pts** A little bit of work
- **- 8 pts** Attempted a constructive count without accounting for changes depending on case
- **- 10 pts** Incorrect with no valid work
- **- 10 pts** Skipped

QUESTION 6

6 Powerset graph 10 / 10

- ✓ **+ 10 pts Correct**

- **10 pts** Skipped

+ **7 pts** [partial credit] Correct, except empty set was ignored.

+ **4 pts** G is 3-regular, with some justification (either a picture or an explanation).

+ **6 pts** Correct sum of weights.

+ **4 pts** [partial credit] Small mistake when finding MST

+ **2 pts** [partial credit] Reasonable but incorrect attempt at drawing graph.

+ **1 pts** [partial credit] Some attempt at finding MST. (e.g. Indicating that you know what a spanning tree is.)

+ **3 pts** [partial credit] Correct minimum spanning tree given incorrect picture.

- **1 pts** Wrong definition of d-regular.

- **10 pts** Incorrect

QUESTION 7

7 Sheffer stroke 10 / 10

✓ **- 0 pts** Correct

- **10 pts** Skipped

- **9 pts** Only considered particular examples

- **1 pts** Did not justify (3)

QUESTION 8

8 Handshake 10 / 10

✓ **- 0 pts** Correct

- **8 pts** Counted orderings instead of combinations

- **9 pts** Incorrect, unclear what is being counted

- **6 pts** double counted all handshakes

- **10 pts** Did not attempt problem

QUESTION 9

9 Hexagon 10 / 10

✓ **- 0 pts** Correct

- **10 pts** Skipped

- **2 pts** Right idea, but need to make the structure of your argument more clear. How exactly are you applying the pigeonhole principle?

- **6 pts** Had the idea of using equilateral triangles. But structure of the argument is wrong, or very unclear.

- **6 pts** Tried the "greedy" approach. But didn't justify correctly with pigeonhole.

QUESTION 10

10 Rigid graph 0 / 10

- **0 pts** Correct

✓ **- 10 pts** Not Graded

- **7 pts** Major flaw in reasoning

- **2 pts** Minor justification needed

QUESTION 11

11 Surjections with small fibers 0 / 10

- **0 pts** Correct

- **2 pts** Does not correctly account for over-counting when dealing with doubly-covered points of Y

- **2 pts** Does surjectivity starting from 20 instead of 30

- **2 pts** Does not account for order of assignments when enforcing surjectivity

- **5 pts** Major error in dealing with doubly-covered points of Y

- **8 pts** Builds matchings which may not be valid functions

- **10 pts** Incorrect without a visible way to adapt towards a correct solution

- **10 pts** Interprets the problem as two separate parts

✓ **- 10 pts** Skipped

QUESTION 12

12 Counting rooted graphs 6 / 10

- **0 pts** Correct

- **10 pts** skipped

- **6 pts** Did not take roots into account

- **4 pts** miscounted number of graphs

- **5 pts** Count trees on fewer than 3 vertices/2 edges

- **7 pts** Counted graphs/rooted trees up to isomorphism, not the number of rooted trees on the given vertices

- **8 pts** Miscounted without a clear argument

- 4 Point adjustment

Miscounted trees - you're right that there's 1 case when the root has degree 2, and then in the case that the root has degree 1, you have 2 remaining choices for the descendant of the root and then you have 1 choice for the descendant of the descendant. Altogether that's 3 + 3x2 = 9 trees.

Name: [redacted]

Student ID#: [redacted]

DO NOT GRADE 10, 11

MATH 61 - FINAL EXAM

0.1. **Instructions.** This is a 180 minute exam. You should feel free to quote any theorems proved in class, as well as anything proved in the homework or discussion section. There are 12 questions—on the exam, you are required to do the first true/false question, and choose 9 of the remaining 11. Only 9 problems other than the true/false question will be graded so *you should indicate which problems you want graded by marking the one you do not want graded with an X*, in the case that you attempt all 6. Each question is worth 10 points. Unless otherwise specified, you are required to justify your answers.

Recall all of our graphs are simple.

Exercise 0.1. Indicate whether the following statements are true or false:

(1) The number of rooted trees on a fixed set of n vertices is n^{n-2}.
(2) Suppose $|X| = n$ and $|Y| = k$. There are n^k many functions from X to Y.
(3) If G has no subgraph isomorphic to $K_{3,3}$ or K_5, then G is planar.
(4) If $G = (V, E)$ is a graph, then the relation R on V, defined by $(x, y) \in R$ if and only if there is a path from x to y in G, is an equivalence relation.
(5) If $T = (V, E)$ is a rooted tree, then the relation D on V, defined $(x, y) \in D$ if x is a descendent of y or if $x = y$, is a partial order.

1) F

2) F

3) T

4) F

5) T

Date: December 9, 2018; Ramsey.

[redacted]

Exercise 0.2. Recall that for X a set, $\mathcal{P}(X)$ denotes the *power set* of X, the set whose elements are the subsets of X: $\mathcal{P}(X) = \{Y : Y \subseteq X\}$. Show that

$$\sum_{i=1}^{n} |\mathcal{P}(\{1, \dots, i\})| = 2^{n+1} - 2.$$

The number of subsets of $\{1, 2, \dots i\}$ can be counted by noting that for each subset, an element of the set is either in that subset or not. So, we have, for a given i,

$$|\mathcal{P}(\{1, \dots i\})| = 2^i.$$

2 choices for each of the i elements; in the subset or not.

Then, we now want to show that

$$\sum_{i=1}^{n} 2^i = 2^{n+1} - 2.$$

We know that $\sum_{i=1}^{n} 2^i = 2^1 + 2^2 + \dots + 2^n$,

which can be rewritten as:

$$= \frac{(2^1 + 2^2 + \dots + 2^n)(2-1)}{(2-1)} = \begin{array}{r} 2^{n+1} + 2^n + \dots + 2^2 \\ - 2^n - \dots - 2^2 - 2^1 \\ \hline \end{array}$$

$$= 2^{n+1} - 2.$$

Because we showed that $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$, we proved that

$$\sum_{i=1}^{n} |\mathcal{P}(\{1, \dots, i\})| = 2^{n+1} - 2. \blacksquare$$

[redacted]

Exercise 0.3. Show that, for all natural numbers n,

$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}.$$

Note: Both algebraic and combinatorial proofs are possible.

1 2 1
1 3 3 1

Base Case: $n=1$.

$$2^1 = \binom{1}{0} + \binom{1}{1}$$

$$2 = 2 \checkmark$$

Induction Step: Assume, for a given k, that $2^k = \binom{k}{0} + \binom{k}{1} + \ldots + \binom{k}{k}$. We now want to show that $2^{k+1} = \binom{k+1}{0} + \binom{k+1}{1} + \ldots \binom{k+1}{k}$.

First, note that $\binom{k}{0} = \binom{k+1}{0} = 1$: $\binom{k}{0} = \frac{k!}{k!\,0!} = 1$. $\binom{k+1}{0} = \frac{(k+1)!}{(k+1)!\,0!} = 1$.

and that $\binom{k}{k} = \binom{k+1}{k+1} = 1$: $\binom{k}{k} = \frac{k!}{k!\,0!} = 1$. $\binom{k+1}{k+1} = \frac{(k+1)!}{(k+1)!\,0!} = 1$.

Now, we multiply the statement $2^k = \binom{k}{0} + \binom{k}{1} + \ldots + \binom{k}{k}$ by 2 to get:

$$2^{k+1} = 2\left(\binom{k}{0} + \binom{k}{1} + \ldots + \binom{k}{k}\right)$$

$$= \binom{k}{0} + \binom{k}{0} + 2\left(\binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{k-1}\right) + \binom{k}{k} + \binom{k}{k}$$

$$= \binom{k+1}{0} + \left[\binom{k}{0} + \binom{k}{1}\right] + \left[\binom{k}{1} + \binom{k}{2}\right] + \ldots + \left[\binom{k}{k-2} + \binom{k}{k-1}\right] + \left[\binom{k}{k-1} + \binom{k}{k}\right] + \binom{k+1}{k+1}.$$

Lemma A:
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

$$\frac{n!}{(n-k)!\,k!} = \frac{(n-1)!}{(n-k)!\,(k-1)!} + \frac{(n-1)!}{(n-k-1)!\,(k)!}$$

$$= \frac{(n-1)!\,(k)}{(n-k)!\,k!} + \frac{(n-1)!\,(n-k)}{(n-k)!\,k!}$$

$$= \frac{n!}{(n-k)!\,k!} \checkmark$$

By Lemma A,

$$= \binom{k+1}{0} + \binom{k+1}{1} + \binom{k+1}{2} + \ldots + \binom{k+1}{k} + \binom{k+1}{k+1}$$

$$= 2^{k+1} \checkmark$$

By mathematical induction, we showed that $2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$.

[redacted]

Exercise 0.4. There are $\binom{52}{5}$ many 5-card hands from a standard 52 card deck. How many of them are either a *straight* or a *flush*? (A *straight* is a sequence of 5 ← not consec? cards whose values are in order, the ace is high (and not low); a *flush* is 5 cards of the same suit).
← e.g. 5 ♡s.

As the ace is high, the cards start with 2. Thus, we have 9 choices for the lowest card of the straight, namely 2 to 10, and 4 choices for each of the suits:

$$9 \cdot 4^5$$

To count the flushes, note that we first have 4 suits to choose from, then, we choose 5 from each of the 13 numbers/symbols:

$$4 \cdot \binom{13}{5}.$$

By Inclusion-Exclusion Principle, we must subtract out the straight flushes: 9 choices for the start number/symbol of the card, and 4 choices for the suit of the cards:

$$9 \cdot 4$$

$$\to 9 \cdot 4^5 + 4 \cdot \binom{13}{5} - 9 \cdot 4$$

[redacted]

Exercise 0.5. How many integer solutions are there to the equation

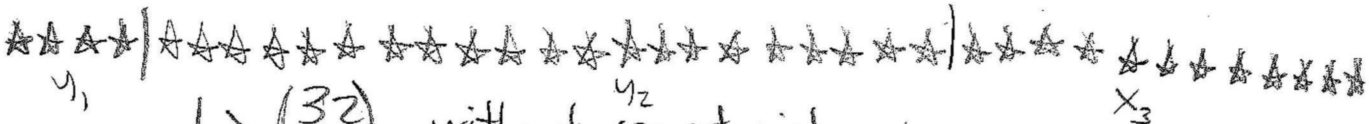
$$x_1 + x_2 + x_3 = 37$$

subject to the constraints that $3 \leq x_1 < 6$, $4 \leq x_2$, and $0 \leq x_3 < 37$.

Let $y_1 = x_1 - 3$ and $y_2 = x_2 - 4$, such that the problem becomes equivalent to finding the number of integer solutions of:

$$y_1 + y_2 + x_3 = 30, \text{ s.t. } 0 \leq y_1 < 3, \; 0 \leq y_2, \; 0 \leq x_3 < 37.$$

We can use stars and bars, based on where to place 2 bars amongst 30 stars:



$\hookrightarrow \binom{32}{2}$. without constraints.

We now must subtract out the cases that don't satisfy the constraints, namely where $y_1 \geq 3$ and $x_3 \geq 37$.

Case 1: where $y_1 \geq 3$.

$\hookrightarrow$ this becomes a problem of choosing 2 bars to place between 27 stars. $\binom{29}{2}$.

Case 2: $x_3 \geq 37$; never satisfied, because then $y_1 + y_2 \leq -7$, yet y_1 and y_2 are greater than 0 by the constraints given. $\times$

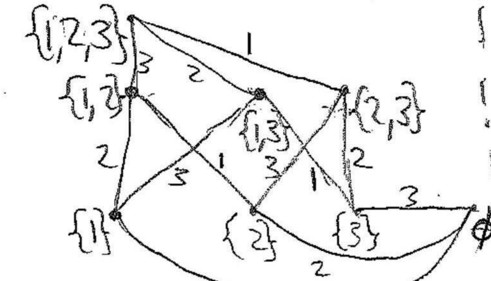
$$\boxed{\binom{32}{2} - \binom{29}{2}}$$

[redacted]

Exercise 0.6. Consider a weighted graph $G = (V, E)$ where $V = \mathcal{P}(\{1,2,3\})$ and v and w are connected by an edge if and only if $|v \Delta w| = 1$, in which case this edge is given weight corresonding to the unique element of $v \Delta w$. In other words, the vertices of G are the subsets of $\{1,2,3\}$ and the edges are pairs of sets where one has exactly one more element than the other. For example, $\{1\}$ and $\{1,2\}$ are connected by an edge, and the weight of this edge is 2.

(1) Is there some d so that G is d-regular?
(2) What is the sum of the weights in a minimal spanning tree?

Yes;
1) $d=3$, as shown on right:

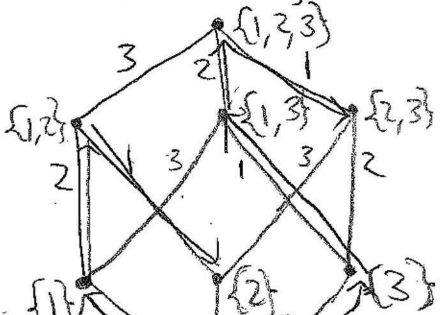


thus, there are 3 choices of a subset Y for $X, Y \subseteq \{1,2,3\}$, $|X \Delta Y| = 1$, as described below, and thus 3 edges that start from a subset X of $\{1,2,3\}$ for any $X \in V$.

Intuitively, this is the case, because for a given subset X of $\{1,2,3\}$, there are only edges to subsets with one more element than what X has, or one less. Thus, this leaves 3 options (recall that an element of a set is either in a given subset of the set, or not):

- add 1 (or remove 1, if $1 \in X$).
- add 2 (or remove 2, if $2 \in X$).
- add 3 (or remove 3, if $3 \in X$).

2) Starting at the empty set vertex, we can use Prim's algorithm:



- we want to choose the weight 1 edge to $\{1\}$, then a weight 2 edge to $\{1,2\}$. Then, we see a weight 1 edge to $\{2\}$ which we can place in our minimal weight spanning tree. From here, there are no 1 or 2-weighted edges; we must pick a 3-weighted edge. ~~To~~ We will choose to pick one to $\{3\}$, and we now have a 1-weighted edge choice to $\{1,3\}$. From here, we have a 2-weighted edge choice to $\{1,2,3\}$ because no one-weighted edges are open until after we head to $\{1,2,3\}$. We pick the one-weighted edge to $\{2,3\}$, and we are done.

$1+2+1+3+1+2+1$
$= 11$

[redacted]

Exercise 0.7. Suppose X is the universal set and for subsets $A \subseteq X$, we write $\overline{A}$ to denote the complement of A in X. The *Sheffer stroke* $\uparrow$ is a single operation on sets defined by $A \uparrow B$ is the complement of $A \cap B$. This has the remarkable property that every other operation on sets may be defined in terms of it.

(1) Show $\overline{A} = (A \uparrow A)$.
(2) Show $A \cup B = (A \uparrow A) \uparrow (B \uparrow B)$
(3) Find a formula for $A \cap B$ using only parentheses and $\uparrow$.

$X \uparrow (A \uparrow B)$
$= X \uparrow \overline{\overline{A \cap B}}$
$= \overline{X \cap \overline{\overline{A \cap B}}}$

Lemma X: $S \cap S = S$ for any S.

Part I: $S \cap S \subseteq S$.
For any arbitrary element $s \in S \cap S$, $s \in S$ and $s \in S$, so $s \in S$ for all $s \in S \cap S \rightarrow S \cap S \subseteq S$.

Part 2: $S \subseteq S \cap S$.
For any arbitrary element $s \in S$, we have that $s \in S$. But, because $s \in S$ and $s \in S$, by definition of the union, $s \in S \cap S$, and thus $s \in S \cap S$ for all $s \in S \rightarrow S \subseteq S \cap S$.

Because $S \cap S \subseteq S$ and $S \subseteq S \cap S$, $S = S \cap S$. ■

1) $A \uparrow A = \overline{(A \cap A)}$ by definition of the Sheffer stroke, but $A \cap A = A$ by Lemma X $\rightarrow A \uparrow A = \overline{A}$ ✓.

2) By Part (1), $A \uparrow A = \overline{A}$ and $B \uparrow B = \overline{B}$, so we want to show that $A \cup B = \overline{A} \uparrow \overline{B} \left(= \overline{(\overline{A} \cap \overline{B})}\right)$

Part 1: $A \cup B \subseteq \overline{(\overline{A} \cap \overline{B})}$.
For any arbitrary element $a \in A \cup B$, $a \in A$ or $a \in B$.
If $a \in A$, $a \notin \overline{A}$, and thus $a \notin \overline{A} \cap \overline{B}$ and so $a \in \overline{\overline{A} \cap \overline{B}}$. If $a \in B$, $a \notin \overline{B}$ and thus $a \notin \overline{A} \cap \overline{B}$, so $a \in \overline{\overline{A} \cap \overline{B}}$. Since this is true for all $a \in A$ or $a \in B \rightarrow$ all $a \in A \cup B$, $A \cup B \subseteq \overline{(\overline{A} \cap \overline{B})}$.

Part 2: $\overline{(\overline{A} \cap \overline{B})} \subseteq A \cup B$.
For any arbitrary element $a \in \overline{(\overline{A} \cap \overline{B})}$, $a \notin \overline{A} \cap \overline{B}$. This means that $a \notin \overline{A}$ or $a \notin \overline{B}$. If $a \notin \overline{A}$, $a \in A$, and thus $a \in A \cup B$. If $a \notin \overline{B}$, $a \in B$, and thus $a \in A \cup B$. Since this is true for all $a \in \overline{(\overline{A} \cap \overline{B})}$, $\overline{(\overline{A} \cap \overline{B})} \subseteq A \cup B$.

Thus, we have shown that, because $\overline{(\overline{A} \cap \overline{B})} \subseteq A \cup B$ and $A \cup B \subseteq \overline{(\overline{A} \cap \overline{B})}$, that $A \cup B = \overline{(\overline{A} \cap \overline{B})} = \overline{A} \uparrow \overline{B}$
$= (A \uparrow A) \uparrow (B \uparrow B)$.

(3) We claim that $A \cap B = (A \uparrow B) \uparrow (A \uparrow B)$.
By part 1, this is equal to $\overline{(A \uparrow B)}$, which in turn is equal to $\overline{\overline{A \cap B}}$. But, the complement of the complement of the set S is S itself (see supplemental page for con't)

[redacted]

I con't.)

To show that $\overline{\overline{S}} = S$ (Lemma Y)

we show that $\overline{\overline{S}} \subseteq S$:

for an arbitrary element $s \in \overline{\overline{S}}$, $s \notin \overline{S}$, so $s \in S$. thus, because we picked an arbitrary s, this is true for all $s \in \overline{\overline{S}}$, so $\overline{\overline{S}} \subseteq S$.

and $S \subseteq \overline{\overline{S}}$.

for an arbitrary element $s \in S$, $s \notin \overline{S}$, and thus $s \in \overline{(\overline{S})}$. So, because we picked an arbitrary $s \in S$, this is true for all $s \in S$, so $S \subseteq \overline{\overline{S}}$,

Thus, because $\overline{\overline{S}} \subseteq S$ and $S \subseteq \overline{\overline{S}}$, $S = \overline{\overline{S}}$.

By Lemma Y, $\overline{\overline{A \cap B}} = A \cap B$.

So, we have shown that $A \cap B = (A \uparrow B) \uparrow (A \uparrow B)$, and we are done.

[redacted]

Exercise 0.8. At the end of finals week there will be a party with 15 guests.

(1) If each person shakes every other person's hand exactly once, how many handshakes have taken place?

(2) Suppose that 7 of the guests have the same cold, so they don't want to shake the hands of the guests who are not sick, but they don't mind shaking each others' hands. If all the sick guests shake hands and all the not-sick guests shake hands, how many hand shakes have taken place?

1) $\binom{15}{2} = \frac{15 \cdot 14}{2} = 105$ ← this is analogous to each pair of 15 people shaking hands exactly once

2) $\binom{7}{2} + \binom{8}{2} = \frac{7 \cdot 6}{2} + \frac{8 \cdot 7}{2} = \frac{7}{2}(6+8) = 49$

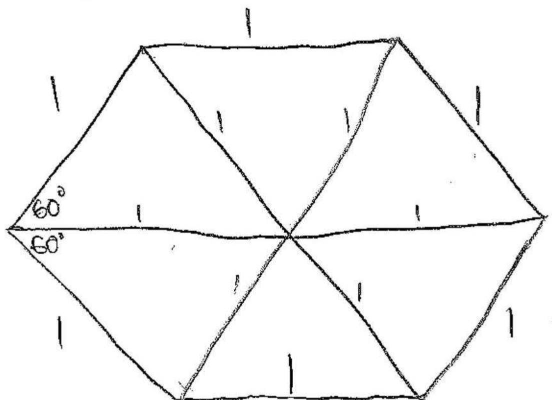
this is analogous to each pair of people in the group of 8 healthy people shaking hands exactly once.

this is analogous to each pair of people in the group of 7 sick people shaking hands exactly once.

[redacted]

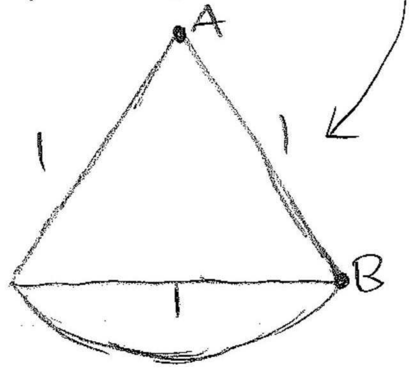
Exercise 0.9. Suppose there are 7 points in the interior of a regular hexagon (i.e. all interior angles are the same) of side length 1. Show that at least two of the points are within distance ~~[illegible]~~ 1 of one another.

Divide the hexagon into 6 equilateral triangles of side length 1:



because the hexagon is regular the triangles will all be equilateral with 60° angles and side length 1.

By the Pigeonhole Principle, 2 points in the triangle fall within the same triangle. the furthest apart they can be within the triangle is distance 1



for a ~~given~~ point at the vertex A, B being more than distance 1 from A means being outside the triangle. And if A isn't at the vertex, then A is less than distance 1 away from all three vertices as well as the points on the triangle's edges.

Thus, at least 2 points of the 7 points in a regular hexagon's interior are within distance 1 of one another.

Exercise 0.10. Recall that a graph G is called *rigid* if the only autorphism of G is the identity function - i.e. the function that sends $v \mapsto v$ and $e \mapsto e$ for every vertex and edge of G respectively. Is there a rigid graph with exactly 3 vertices? Give an example or show there is no such graph.

Exercise 0.11. Suppose X is a set with 30 elements and Y is a set with 20 elements. How many functions $f : X \to Y$ are there satisfying both of the following properties (at the same time):

(1) $f : X \to Y$ is onto (i.e. surjective) ($\to$ all $y \in Y$ is used up).
(2) For every $y \in Y$, $|\{x \in X : f(x) = y\}| \leq 2$.

[redacted]

Exercise 0.12. How many rooted trees are there on the vertices $\{x_1, x_2, x_3\}$? Are there more or fewer than the total number of graphs on these vertices?

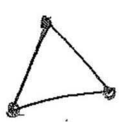
$3 \cdot 2 = 6$

choices for root

Case 1: remaining 2 are children of the root

Case 2: a tree of one vertex per level, i.e. .

This is <u>fewer</u> than the total number of graphs on these vertices, because there are 8:



only one choice w/ 3 edges, and w/ 0 edges.

For the 1-edge / 2-edge cases, you have 3 choices each about which pair of vertices to connect (1-edge) or, not connect (2 edges).

$1+3+3+1 = 8$ graphs.

$6 < 8$