

MIDTERM 2 (MATH 61, SPRING 2015)

Your Name: _____

UCLA id: _____

Math 61 Section: _____

Date: _____

The rules:

You MUST simplify completely and BOX all answers with an **INK PEN**.

You are allowed to use only this paper and pen/pencil. No calculators.

No books, no notebooks, no web access. You MUST write your name and UCLA id.

Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: those caught writing after time get automatic 10% score deduction.

Points:

1 |

2 |

3 |

4 |

5 |

Total:

(out of 100)

Problem 1. (20 points)

Compute the number of (shortest) grid walks from $(0, 0)$ to $(9, 9)$ which:

- a) do not go through any of the other diagonal points $(1, 1), (2, 2), \dots, (8, 8)$
- b) stay on or above $y = x - 1$ diagonal
- c) stay on or above $y = x$ diagonal AND do not go through $(6, 6)$
- d) stay on or above $y = x$ diagonal AND on or below $y = x + 1$ diagonal.

Note: You can use the formula for Catalan numbers or write the answer in terms of C_n 's.

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Problem 2. (20 points)

Compute the number of subgraphs of G isomorphic to H , where

a) $G = K_{7,9}$, $H = C_4$

b) $G = K_{7,9}$, $H = P_4$

c) $G = C_9$, $H = P_4$

d) $G = K_9$, $H = K_{2,3}$

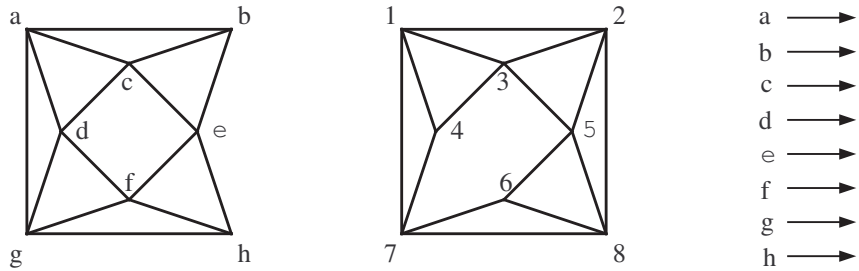
Problem 3. (15 points)

Let $a_1 = 2$, $a_2 = 7$, $a_{n+1} = a_n + 2a_{n-1}$. Solve this LHR and find a closed formula for a_n .

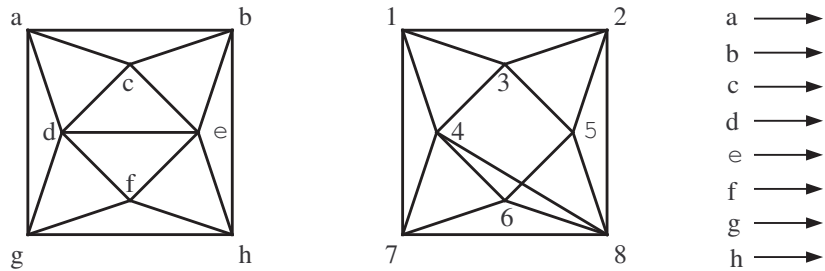
Problem 4. (15 points)

Decide whether the following pairs of graphs on 8 vertices are isomorphic or non-isomorphic.

a)



b)



Important: In case of isomorphism, you must write a bijection in the figure above (in ink). No need for further arguments. In case non-isomorphism, you must say so and present an argument why two graphs are not isomorphic.

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink. No explanation is required or will be considered.

- T F** (1) Isomorphic graphs have the same number of edges.
- T F** (2) Isomorphic graphs have the same number of connected components.
- T F** (3) Isomorphic graphs have the same number of 4-cycles.
- T F** (4) $F_n \leq C_n$ for all integer n .
- T F** (5) Sequence $(3, 3, 3, 3, 3)$ is a valid score of a simple graph.
- T F** (6) Sequence $(4, 4, 4, 4, 2)$ is a valid score of a simple graph.
- T F** (7) Sequence $(4, 4, 4, 2, 2)$ is a valid score of a simple graph.
- T F** (8) Sequence $(4, 4, 2, 2, 2)$ is a valid score of a simple graph.
- T F** (9) Sequence $(2, 2, 2, 0, 0)$ is a valid score of a simple graph.
- T F** (10) Graph C_8 is a subgraph of $K_{7,7}$.
- T F** (11) Graph C_8 is a subgraph of $K_{9,3}$.
- T F** (12) Graph P_8 is a subgraph of $K_{9,3}$.
- T F** (13) Graph K_4 is a subgraph of $K_{7,7}$.
- T F** (14) Graph K_9 has 72 edges.
- T F** (15) Catalan numbers modulo 2 are periodic with period 6.

Problem 1. (20 points)

Compute the number of (shortest) grid walks from $(0, 0)$ to $(9, 9)$ which:

a) do not go through any of the other diagonal points $(1, 1), (2, 2), \dots, (8, 8)$.

Solution: If the first step of a good walk is up, the entire walk remains on or above the first super diagonal (that is, $(0, 1), (1, 2), \dots, (8, 9)$). Therefore the set

{good walks whose first step is up}

is in bijection with the set

{on-or-above-the-diagonal walks on an 8×8 grid}

so there are C_8 of them. Similarly, the good walks whose first step is right are in bijection with the on-or-below-the-diagonal walks on an 8×8 grid so there are C_8 good walks in this case.

Answer: $2C_8$.

b) stay on or above $y = x - 1$ diagonal

Solution: The good walks are in bijection with the on-or-above-the-diagonal walks on the 10×10 grid with corners $(0, -1), (0, 9), (10, 9), (10, -1)$. The bijection takes a good grid walk (on or above $y = x - 1$) on the 9×9 grid and on the segment from $(0, -1)$ to $(0, 0)$ and the segment from $(9, 9)$ to $(10, 9)$.

Answer: C_{10} .

c) stay on or above $y = x$ diagonal AND do not go through $(6, 6)$

Solution: The number of walks that **do** go through $(6, 6)$ is C_6C_3 because they are fully determined by (independently) choosing an on-or-above-the-diagonal walk from $(0, 0)$ to $(6, 6)$ and an on-or-above-the-diagonal walk from $(6, 6)$ to $(9, 9)$. Because walks that **do not** go through $(6, 6)$ are the complement of those that do, we simply subtract C_6C_3 from the total number of on-or-above-the-diagonal walks on the 9×9 grid.

Answer: $C_9 - C_6C_3$.

d) stay on or above $y = x$ diagonal AND on or below $y = x + 1$ diagonal.

Solution: To stay on or above $y = x$, the first step must be up. Then to stay on or below $y = x + 1$, the second step must be right. Following this pattern, each odd step is up and each even step is right so there is only one such walk.

Answer: 1.

Problem 2. (20 points)

Compute the number of subgraphs of G isomorphic to H , where

a) $G = K_{7,9}$, $H = C_4$.

Solution: Because each edge in G goes between the two sides, any graph isomorphic to H must have 2 vertices from each side, i.e. $\binom{7}{2}\binom{9}{2}$ choices of vertices. After deleting all other vertices (and their edges), each of the vertices in our subgraph has degree 2 and thus it is a C_4 so we have no more decisions to make.

Answer: $\binom{7}{2}\binom{9}{2}$.

b) $G = K_{7,9}$, $H = P_4$.

Solution: Again we must choose 2 vertices from each side, for which we again have $\binom{7}{2}\binom{9}{2}$. However, P_4 has only 3 edges so we must pick which of the 4 edges in our subgraph to delete.

Answer: $4\binom{7}{2}\binom{9}{2} = 9 * 8 * 7 * 6$

c) $G = C_9$, $H = P_4$.

Solution: A P_4 is chosen by picking 3 edges in a row in G . Because G has 9 edges each with an edge on either side this leaves 9 choices for the middle edge of H , fully determining H .

Answer: 9.

d) $G = K_9$, $H = K_{2,3}$

Solution: First we must choose 2 of the 9 vertices in G to be on “Side A” of H , i.e. $\binom{9}{2}$ choices. Then we (independently) choose 3 from the remaining 7 vertices to be on “Side B” of H , i.e. $\binom{7}{3}$ choices. To make H isomorphic to $K_{2,3}$ we must delete exactly the other 4 vertices (and their edges) as well as all of the edges within Side A and all of the edges within Side B. Therefore H is fully determined

Answer: $\binom{9}{2}\binom{7}{3} = \frac{9!}{4!3!2!} = \binom{9}{2,3}$.

Problem 3. (15 points)

Let $a_1 = 2$, $a_2 = 7$, $a_{n+1} = a_n + 2a_{n-1}$. Solve this LHR and find a closed formula for a_n .

Solution: Because $x^2 - x - 2 = (x + 1)(x - 2)$ has roots -1 and 2 , Theorem 7.2.11 in the textbook says any solution to the LHR must be of the form $a_n = b(-1)^n + d(2^n)$. To find b and d , we solve the system

$$2 = a_1 = -b + 2d$$

$$7 = a_2 = b + 4d$$

to get $d = \frac{3}{2}$, $b = 1$.

$$\text{Answer: } a_n = (-1)^n + \frac{3}{2}(2^n) = (-1)^n + 3(2^{n-1})$$

Problem 4. (15 points)

Decide whether the following pairs of graphs on 8 vertices are isomorphic or non-isomorphic. We will call the left graph G and the right graph H in parts (a) and (b).

a) Because $b, h, 4,$ and 6 are the only vertices with degree three, we know either $b \rightarrow 4, h \rightarrow 6$ or $b \rightarrow 6, h \rightarrow 4$. Because G is horizontally symmetric, either one is fine so we'll say $b \rightarrow 4, h \rightarrow 6$. The only shared neighbor of b and h is e so e must map to the only shared neighbor of 4 and 6 , i.e. 7 . b and e both neighbor c so c must map to the unique shared neighbor of 4 and 7 , i.e. 1 . Exchanging b for h in the previous sentence shows f maps to 8 . The only shared neighbor of c and f that hasn't been labelled is d so, using the same logic as before, d must map to 2 . Since a neighbors b , a must map to a neighbor of 4 so 3 is the only remaining possibility. This leaves only 5 for g to map to.

Answer:

$$a \rightarrow 3$$

$$b \rightarrow 4$$

$$c \rightarrow 1$$

$$d \rightarrow 2$$

$$e \rightarrow 7$$

$$f \rightarrow 8$$

$$g \rightarrow 5$$

$$h \rightarrow 6$$

b) G and H cannot be isomorphic because H contains a subgraph isomorphic to K_4 (the subgraph has vertex set $\{4, 6, 7, 8\}$) but G does not.

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink. No explanation is required or will be considered.

- (1) Isomorphic graphs have the same number of edges. **True.**
- (2) Isomorphic graphs have the same number of connected components. **True.**
- (3) Isomorphic graphs have the same number of 4-cycles. **True.**
- (4) $F_n \leq C_n$ for all integer n . **True.**

Explanation: This can be proven by induction. Using the convention $C_0 = F_0 = 1$, the base cases (for indices 0 and 1) are $1 \leq 1$ and $1 \leq 1$. The induction hypothesis will be that for all $k < n$ (here $n > 2$) $F_k \leq C_k$. Then

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k} \geq C_{n-1} + C_{n-2} \geq F_{n-1} + F_{n-2} = F_n.$$

- (5) Sequence (3, 3, 3, 3, 3) is a valid score of a simple graph. **False.**

Explanation: By the Handshake Theorem the sum of the vertices' degrees must be even for any graph as opposed to 15 in this case.

- (6) Sequence (4, 4, 4, 4, 2) is a valid score of a simple graph. **False.**

Explanation: If a simple graph has 5 vertices, any degree 4 vertex neighbors all other vertices so the degree 2 vertex would have to neighbor all the degree 4 vertices. Since there are 4 of them it could not have degree 2.

- (7) Sequence (4, 4, 4, 2, 2) is a valid score of a simple graph. **False.**

Explanation: Same as problem (6).

- (8) Sequence (4, 4, 2, 2, 2) is a valid score of a simple graph. **True.**

Explanation: We can build such a graph by making a C_4 with vertex set $\{a, b, c, d\}$, adding a vertex e , and connecting e to vertices b and c . Like a cartoon house.

- (9) Sequence (2, 2, 2, 0, 0) is a valid score of a simple graph. **True.**

Explanation: There is exactly one such graph. We build it by taking a C_3 with vertex set $\{a, b, c\}$ and adding two vertices d and e who have no neighbors. Like a smiley face with a triangle for a mouth.

- (10) Graph C_8 is a subgraph of $K_{7,7}$. **True.**

Explanation: If one side of $K_{7,7}$ is labelled with letters a, \dots, g and the other is labelled with numbers $1, \dots, 7$ then we can find a C_8 by deleting vertices $e, f, g, 5, 6, 7$ (with their edges) and deleting all edges except the cycle $a, 1, b, 2, c, 3, d, 4, a$.

- (11) Graph C_8 is a subgraph of $K_{9,3}$. **False.**

Explanation: There must be exactly 4 vertices from each side so the 3 vertex side cannot provide enough.

- (12) Graph P_8 is a subgraph of $K_{9,3}$. **False.**

Explanation: Same as problem (11).

(13) Graph K_4 is a subgraph of $K_{7,7}$. **False.**

Explanation: Any subgraph of a bipartite graph is bipartite however $K_{7,7}$ is while K_4 is not.

(14) Graph K_9 has 72 edges. **False.**

Explanation: K_9 has $\binom{9}{2} = 36$ edges.

(15) Catalan numbers modulo 2 are periodic with period 6. **False.**

Explanation: If this were true for any $n \geq 1$, $C_n = C_{n+6} \pmod{2}$. However $C_3 = 5$ while $C_9 = 4862$.