MIDTERM 1 (MATH 61, SPRING 2017)

Your Name: ____

UCLA id: ____

Math 61 Section: ___________

Date: ______________________

The rules:
You MUST simplify completely and BOX all answers with an **INK PEN**.
You are allowed to use only this paper and pen/pencil. No calculators.
No books, no notebooks, no web access. You MUST write your name and UCLA id.
Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Warning: those caught violating the rules get automatic 10% score deduction.

Points:

1	
2	
3	
4	
5	

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Total: **(out of 100)**

Problem 1. (20 points)

Compute the probability that 4-subset A of $\{1, 2, \ldots, 10\}$ satisfies:

a) A has no odd numbers,

b) A has at least one number ≤ 3,

c) A contains 1 but not 7.

d) the smallest number in A is divisible by 3

Solutions.

a) The 4-subsets of $\{1, \ldots, 10\}$ with no odd numbers are the 4-subsets of $\{2, 4, 6, 8, 10\}$ which has 5 elements. Therefore the probability is $\frac{\binom{5}{4}}{\binom{10}{4}} = \frac{1}{42}$.

b) The 4-subsets of $\{1, \ldots, 10\}$ with no number ≤ 3 are the 4-subsets of $\{4, \ldots, 10\}$ which has 7 elements. Therefore the probability is $1 - \frac{\binom{7}{4}}{\binom{10}{4}} = \frac{5}{6}$.

c) The 4-subsets of $\{1, \ldots, 10\}$ containing 1 but not 7 can be expressed in one way as $\{1\} \cup S$ where S is a 3-subset of $\{1, 2, 4, 5, 6, 8, 9, 10\}$ which has 8 elements. The probability must then be $\frac{\binom{8}{3}}{\binom{10}{4}} = \frac{4}{15}$.

d) The 4-subsets of $\{1, \ldots, 10\}$ with smallest number divisible by 3 must have smallest number 3 or 6. Here 9 is not an option because there are fewer than 4 numbers ≥ 9 in the set. The 4-subsets of $\{1, \ldots, 10\}$ with smallest number 3 can be written in exactly one way as $\{3\} \cup S$ where S is a 3-subset of $\{4, 5, 6, 7, 8, 9, 10\}$ which has 7 elements and each such S makes $\{3\} \cup S$ a 4-subset with smallest number 3. There are then $\binom{7}{3}$ 4-subsets with smallest number 3. By similar reasoning there are $\binom{4}{3}$ subsets with smallest number 6. It follows that the probability is $\frac{\binom{7}{3} + \binom{4}{3}}{\binom{10}{4}} = \frac{13}{70}$.

Problem 2. (20 points)

Let $X = \mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all non-negative integers. For each of the following functions $f : X \to X$ decide whether they are injective, surjective, bijective:

a) $f(x) = x + 1$

b) $f(x) = x^2 - 1$

c) $f(x) = 2x$

d) $f(x) = (x^2 + 2x)/(x + 2)$

Solutions.

a) That f is injective but not surjective is a fundamental properties of $\mathbb{N}$. If $x, y \in \mathbb{N}$ and $f(x) = f(y)$, then $x + 1 = y + 1$ and so $x = y$. This shows f is injective. Since $x + 1 \neq 0$ for all $x \in \mathbb{N}$, f is not surjective and consequently is not bijective.

b) There is no $y \in X$ so that $y = 0^2 - 1$. Therefore it is not the case that f is a function $X \to X$.

c) Suppose $x, y \in X$ and $f(x) = f(y)$. Then we would have $2x = 2y$ which gives us $x = y$. That $x = y$ follows from our assumptions shows that f is injective. However f is not surjective since $f(x) = 2x$ is an even integer whenever $x \in X = \mathbb{N}$ and so $f(x)$ cannot be an odd element of $\mathbb{N}$. For example, $f(x) \neq 1 \in X$ for every $x \in X$. Therefore f is injective, but not surjective or bijective.

d) We have that for every $x \in \mathbb{N}$, $f(x) = (x^2 + 2x)/(x + 2) = x(x + 2)/(x + 2) = x$. Then f is invertible, being the identity map on X and therefore injective, surjective, and bijective.

Problem 3. (15 points)

Let $a_n = 1111\cdots 1$ (n ones). Suppose a_k is divisible by 97. Use induction to show that $a_{k\cdot n} = 0 \bmod 97$, for all $n \geq 1$.

Solution.

BASE: For $n = 1$, we have $a_{k\cdot 1} = a_k = a_k - 0$ is divisable by 97. Therefore $a_k = 0 \bmod 97$.

STEP: Suppose n was a particular natural number with the property that $a_{k\cdot n} = 0 \bmod 97$. By the definition of the sequence $a_1, a_2, \ldots$ and interpreting the base 10 expansion we have $a_{k\cdot(n+1)} = a_k + 10^k a_{k\cdot n}$. By induction hypothesis and since 10^k is an integer, the product $10^k a_{k\cdot n} = 0 \bmod 97$. Recall that $a_k = 0 \bmod 97$ (from the base case). Then the sum $a_{k\cdot(n+1)} = 0 + 0 = 0 \bmod 97$. This completes the induction.

Problem 4. (15 points)

Find closed formulas for the following sequences :

a) $4, 4, 6, 8, 12, 18, 28, 44, 70, 112, \ldots$

b) $a_1 = 1,\ a_{n+1} = a_n \cdot \binom{n+1}{2}$

c) $a_1 = 1,\ a_2 = 1,\ a_{n+1} = a_{n-1} - a_n$ for $n \geq 2$.

Note: you can express a_n in terms of Fibonacci numbers F_n.

Solutions. In each case, either of the following formulas is correct.

a) We have:

$$a_n \;=\; 2F_n + 2 \;=\; 4 \;+\; 2\big(F_1 + F_2 + \ldots + F_{n-2}\big)$$

b) We have:

$$a_n \;=\; \frac{(n-1)!^2 n}{2^{n-1}} \;=\; \frac{(n-1)!\, n!}{2^{n-1}}$$

c) We have: $a_n \;=\; (-1)^n F_{n-3}$ for $n \geq 4$, and $a_1 = a_2 = 1$ and $a_3 = 0$.

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink. No explanation is required or will be considered.

T **F** (1) The number of functions from $\{A, B, C, D\}$ to $\{1, 2, 3\}$ is equal to 4^3.

T **F** (2) The sequence $1, 3/2, 5/3, 7/6, 9/8, \ldots$ is increasing.

T **F** (3) The sequence $-1, -2, -3, -4, \ldots$ is non-increasing.

T **F** (4) There are 4 anagrams of the word MAMA.

T **F** (5) There are infinitely many Fibonacci numbers which are divisible by 3.

T **F** (6) The number of permutations of $\{1, 2, 3, 4, 5\}$ is smaller than 123.

T **F** (7) The number of 3-permutations of $\{1, 2, 3, 4, 5, 6\}$ is equal to $\binom{6}{3}$.

T **F** (8) The number of 3-subsets of $\{1, 2, 3, 4\}$ is equal to 4.

T **F** (9) The number of permutations of $\{1, 2, \ldots, n\}$ which have n preceding $n-1$ (not necessarily immediately) is equal to $n!/2$

T **F** (10) For every $A, B \subset \{1, 2, \ldots, 12\}$ we have $|A \cap B| < |A \cup B|$.

T **F** (11) For all $n \geq 1$, we have

$$\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \ldots + \binom{2n}{2n} = 2^{2n-1}.$$

T **F** (12) The number of grid walks from $(0, 0)$ to $(10, 10)$ going through $(3, 7)$ is equal to $\binom{10}{3}^2$.

T **F** (13) The number of grid walks from $(0, 0)$ to $(10, 10)$ avoiding $(10, 0)$ and $(0, 10)$ is equal to $\frac{1}{2}\binom{20}{10}$.

T **F** (14) The number of anagrams of MISSISSIPPI which begin with M is greater than the number of anagrams which begin with S.

T **F** (15) The following parabolas are drawn in the plane:

$$y = x^2 - n^2 x - n^3, \qquad n = 1, \ldots, 12.$$

Then the regions of the plane separated by these parabolas can be colored with two colors in such a way that no two adjacent regions have the same color.

Solutions. FFTFT TFTTF TTFFT