MIDTERM 1   (MATH 61, SPRING 2015)

Your Name: _____

UCLA id: _____

Math 61 Section: _____

Date: _____

The rules:

You MUST simplify completely and BOX all answers with an **INK PEN**.
You are allowed to use only this paper and pen/pencil. No calculators.
No books, no notebooks, no web access. You MUST write your name and UCLA id.
Except for the last problem, you MUST write out your logical reasoning and/or
proof in full. You have exactly 50 minutes.

Warning: those caught writing after time get automatic 10% score deduction.

Points:

1 |

2 |

3 |

4 |

5 |

_____

Total: **(out of 100)**

Problem 1.   (20 points)

Compute the number of permutations $(x_1, x_2, \ldots, x_n)$ of $\{1, 2, \ldots, 9\}$ such that:

a) $x_1 = 2$,

b) $x_1 \cdot x_2 \cdot x_3 = 6$,

c) $x_1 = x_2 = x_3 \bmod 7$,

d) $x_1 < x_2 < 5$.

Solutions.

a) Since there are 8 other elements, the number should be 8!.

b) The first three numbers should be $1, 2, 3$ in some order, so the number of permutation should be 3!6!.

c) Since we don't have three numbers from the set such that they are all the same mod 7, the answer should be 0.

d) If $x_2 = 4$, $3 \cdot 7!$. If $x_2 = 3$, $2 \cdot 7!$. If $x_2 = 2$, $3 \cdot 7!$. So the answer is the sum which is $6 \cdot 7!$.

Problem 2. (20 points)

Let $X = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ be the set of all integers. For each of these relations R, decide whether they are reflexive, symmetric or transitive (or neither).

a) xRy if and only if $|x| = |y|$.

b) xRy if and only if $x + 2y = 0 \bmod 3$.

c) xRy if and only if $x^2 + 2y^2 = 0 \bmod 3$.

d) xRy if and only if $x^3 + 122y^3 = 0 \bmod 3$.

Solutions.

a) $|x| = |x|$ for all x, so reflexive.
 $xRy \iff |x| = |y| \iff |y| = |x| \iff yRx$, so symmetric.
 If $|x| = |y|$ and $|y| = |z|$, then $|x| = |z|$, so transitive.

b) $x + 2x = 3x = 0 \bmod 3$. So reflextive
 $xRy \iff x + 2y = 0 \bmod 3 \iff x = y \bmod 3 \iff y + 2x = 0 \bmod 3 \iff yRx$. So symmetric.
 If xRy and yRz, then $x = y \bmod 3$, and $y = z \bmod 3$. So $x = z \bmod 3$, which means the relation is transitive.

c) Similar to the previous problem, the relation is reflexive, symmetric and transitive.

d) Similar to the previous two problems, the relation is reflextive, symmetric and transitive.

Problem 3.   (15 points)

Let $A = (0,0)$, $B = (10,10)$. Find the number of (shortest) grid walks γ from A to B, such that:

a) γ never visits points $(0,10)$, $(10,1)$, $(5,5)$.

b) γ visits all points $(1,1)$, $(2,2)$, $(3,3)$, ..., $(9,9)$.

c) γ visits points $(5,0)$ and $(5,10)$, but not $(5,5)$.

Solutions.

a) $\binom{20}{10} - \binom{10}{0}\binom{10}{10} - \binom{11}{10}\binom{9}{0} - \binom{10}{5}\binom{10}{5}$.

b) 2^{10}.

c) Since we can only go to the left or up on a grid walk, every grid walk that visits (5,0) and (5,10) must also visit (5,5). So the answer should be 0.

Problem 4. (15 points)

Recall the Fibonacci sequence: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, etc.

Prove that $F_n \leq 2^{n-1}$.

Solution:

Base step:
$F_1 = 1 \leq 2^0$; $F_2 = 1 \leq 2^1$.
Induction step:
Assume that $F_{n-1} \leq 2^{n-2}$ and $F_n \leq 2^{n-1}$, then $F_{n+1} = F_n + F_{n-1} \leq 2^{n-1} + 2^{n-2} \leq 2^n$.

Problem 5.   (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answers with ink.   No explanation is required or will be considered.

T   **F**   (1)   The number of functions from $\{A, B, C, D\}$ to $\{1, 2, 3\}$ is equal to 4^3.

T   **F**   (2)   The sequence $10, 21, 32, 43, \ldots$ is increasing.

T   **F**   (3)   The sequence $2/1$, $3/2$, $4/3$, $5/4$ is non-increasing.

T   **F**   (4)   There are 20 anagrams of the word $BUBUB$.

T   **F**   (5)   There are more anagrams of the words $AAAACCC$ which
begin with A than with C.

T   **F**   (6)   There are infinitely many Fibonacci numbers $= 1$ mod 3.

T   **F**   (7)   There are infinitely many binomial coefficients $\binom{n}{k} = 1$ mod 17.

T   **F**   (8)   Each of the 14 students wrote on a paper 10 distinct numbers, from the
set $\{1, 2, \ldots, 100\}$. Then there are two students who have at least 2 numbers
in common on their lists.

T   **F**   (9)   The probability that a random 10-subset of $\{1, 2, \ldots, 19\}$
contains 10 is equal to $1/2$.

T   **F**   (10)   For every two subsets $A, B \subset U$, we must have $|A \smallsetminus B| = |B \smallsetminus A|$.

T   **F**   (11)   For every two subsets $A, B \subset U$, we must have $|A \cup B| \geq |\overline{B}|$

T   **F**   (12)   Every surjection that is also a bijection must be also an injection.

T   **F**   (13)   Every surjection that is also an injection must be also a bijection.

T   **F**   (14)   Let $\mathcal{A}$ be the set of 3-subsets of $[9] = \{1, 2, \ldots, 9\}$. Similarly, let $\mathcal{B}$
be the set of 6-subsets of $[9]$. Consider a map $f : A \to [9] \smallsetminus A$. Then f
is a bijection from $\mathcal{A}$ to $\mathcal{B}$.

T   **F**   (15)   The pigeon hole principle was proved in class by induction.

Solutions: F T T F T T T F F F F T T T F.

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