MIDTERM 1 (MATH 61, FALL 2013)

Your Name: ____________________

UCLA id: ____________________

Math 61 Section: ________

Date: ____________

The rules:
You MUST simplify completely and BOX all answers with an INK PEN.
You are allowed to use only this paper and pen/pencil. No calculators.
No books, no notebooks, no web access. You MUST write your name and UCLA id.
Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Class Statistics
High: 98 (98%)
Low: 25 (25%)
Median: 76 (76%)
Mean: 72.2 (72%)

Points:

1	20
2	18
3	15
4	15
5	28

Total: 96 (out of 100)

Problem 1. (20 points)

Compute the number of 3-subsets A of $\{1, 2, \ldots, 9\}$ such that:

a) A has no odd numbers,

b) A has at least one number ≤ 3,

c) the smallest number in A is divisible by 3.

d) the sum of numbers in A is exactly 7.

(a) All even

2, 4, 6, 8

$\boxed{\binom{4}{3}}$

(b) No # ≤ 3 : 4, 5, 6, 7, 8, 9

$\rightarrow \binom{6}{3}$

All # : 1 – 9

$\binom{9}{3}$

$\boxed{\binom{9}{3} - \binom{6}{3}}$

(c) Small is divisible by 3

– no 1, 2 ; has 3 ; 4, 5, 6, 7, 8, 9

– no 1, 2, 3, 4, 5 ; has 6 ; 7, 8, 9

$\boxed{\binom{6}{2} + \binom{3}{2}}$

(d) 1+2+3 = 6

1+2+4 = 7

1+3+3 ← repeated

only one combination

$\boxed{1}$

$$a_5 = 3\left[25 - \frac{6\cdot 5}{2}\right] + 1$$
$$= 3(10) + 1 = 31 \checkmark$$

$25 - 15$

(c) $a_1 = 1$ $\qquad 1 = \binom{2}{2}$

$a_2 = 1\binom{3}{2}$

$a_3 = 1\binom{3}{2}\binom{4}{2}$

$a_8 = 2 + \sum_{k=1}^{6} = \cancel{2+1+1+2+3+5+8} = 22 \checkmark$ $\qquad \frac{2!}{1\cdot 2^1} = 1$

Problem 2. (20 points)

Find closed formulas for the following sequences :

a) 1, 4, 10, 19, 31, 46, 64, 85, 109, . . . $\qquad a_8 = 22$

b) 2, 2, 3, 4, 6, 9, 14, 22, 35, 56, . . .

c) $a_1 = 1,\ a_{n+1} = a_n \cdot \binom{n+1}{2}$

d) $a_1 = 1,\ a_2 = 1,\ a_{n+1} = a_{n-1} - a_n$ for $n \geq 2$.

Note: you can express a_n in terms of Fibonacci numbers F_n.

$a_n = \binom{n+1}{2}\binom{n}{2}\cdots\binom{2}{2}$

$\frac{(n+1)!}{2!(n-1)!} \cdot \frac{n!}{2!(n-2)!} \cdot \frac{(n-1)!}{2!(n-3)!} \cdot \frac{2!}{2}$

(a) 1 4 10 19 31 46 64 85

3 6 9 12 15 18 21

n = 1 2 3 4 5 6

(c) $\boxed{a_n = \frac{(n+1)!}{(n-1)!\,2^n}}$ $\boxed{= \frac{\cancel{n^2+n}}{2^n}}$

18

$a_n = 3(n-1) + a_{n-1} + 1$

$a_n = 3(n-1) + 3(n-2) + a_{n-2} + 1$

$a_n = 3[(n-1) + (n-2) + (n-3) + \cdots (n-n)] + 1$

(d) $a_1 = 1$

$a_2 = 1$

$a_3 = a_1 - a_2 = 1 - 1 = 0$

$a_4 = a_2 - a_3 = 1$

$a_5 = a_3 - a_4 = -1$

$a_6 = a_4 - a_5 = 2$

$a_7 = a_5 - a_6 = -3$

$a_8 = a_6 - a_7 = 5$

n = 1 2 3 4 5 6 7 8

1, 1, 0, 1, −1, 2, −3, 5, ...

F_n = 1, 1, 2, 3, 5

n = 1 2 3 4 5

(a) $\boxed{a_n = 3\left[n^2 - \frac{(1+n)n}{2}\right] + 1}$ ✓

n = 5

$6 = 2 + 4 = 2 + (1+3)$

$= 2 + (1+1+2)$

n = 6

$F_n \to 5$

3

$1 \to 2$

(b) 2, 2, 3, 4, 6, 9, 14, 22, 35

1 1 2 3 5 8 13

1 2 3 4 5 6 7 8 9

1 1 2 3 5

(b) $\boxed{\begin{array}{l} a_n = 2 + \sum_{k=1}^{n-2} F_k \text{ for } n \geq 3 \\ a_n = 2 \text{ for } n = 1 \\ a_n = 2 \text{ for } n = 2 \end{array}}$

$2 + \sum^{2} \quad 2 + 1$

$1 \to 3$

$a_n = 4 = 1 + 1 + 1 + 2$

Check: $a_5 = \binom{6}{2}\binom{5}{2}\binom{4}{2}\binom{3}{2}\binom{2}{2}$

$a_n = \frac{(n+1)!}{2!(n-1)!} \cdot \frac{n!}{2!(n-2)!} \cdot \frac{(n-1)!}{2!(n-3)!} \cdot \frac{(n-2)!}{2!(n-4)!} \cdots \frac{2\cdot 1}{2\cdot 2}$ $\qquad \frac{n-3}{2!(n-}$

$\frac{(n+1)!}{(n-1)!\,2^n} = \frac{(n+1)(n)}{2^n} = \frac{n^2+n}{2^n}$

(d) $\boxed{a_n = \begin{cases} F_{n-3}(-1)^n \text{ for } n \geq 4; \\ 1 \text{ for } n = 1 \\ 1 \text{ for } n = 2 \\ 0 \text{ for } n = 3 \end{cases}}$ ✓

P-3

Problem 3. (15 points)

Let $a_n = 1111\cdots1$ (n ones). Suppose a_k is divisible by 97. Use induction to show that $a_{k\cdot n} = 0 \bmod 97$, for all $n \geq 1$.

① Base case $n = 1$

$a_{k\cdot 1} = a_k = 0 \bmod 97$

② Inductive steps

Assume $a_{kn} = 0 \bmod 97$ holds.

$a_{k(n+1)} = a_{kn+k} = \underbrace{11111}_{kn}\cdots\underbrace{1}_{k}$

$\therefore a_{kn+k} = a_{kn} \cdot 10^c + a_k$ for some positive integer $c = k$

$a_{kn+k} = (a_{kn} \cdot 10^c + a_k) \bmod 97$

$= (a_{kn} \cdot 10^c) \bmod 97 + a_k \bmod 97$

($\because$ 10 and 97 has no common factor)

$= 10^c \, a_{kn} \bmod 97 + a_k \bmod 97$

$= 0 \bmod 97$

$a_{k(n+1)} = 0 \bmod 97$

$\therefore$ It is proved by induction.

15

Problem 4. (15 points)

There are 6 candidates in a student election, to form a 3-student committee. The total number of students is 105. Each student votes for exactly three candidates (of the 6). Prove that at least one candidate had 53 votes.

Pigeonhole Principle and proof by contradiction.

Total # of votes = (105 students) (3 votes / students) = 315 votes

Suppose at most one candidate had 52 votes.

Largest possible # of votes = (52 votes / candidate) (6 candidates)

= 312 votes

∵ 312 < 315

Largest possible # < Total # of votes

∴ Contradiction ✓

∴ At least one candidate had 53 votes.

15

x y y z x z

1 0 0 1 1 1

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

1+15+15+1 . 32

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

Circle correct answer with ink. No explanation required. $|Y|^{|X|}$

T F (1) The number of functions from $\{A, B, C\}$ to $\{1, 2, 3, 4, 5\}$ is equal to 15.

~~T~~ F (2) The relation R on all integers $\mathbb{Z}$ is defined by xRy if and only if $x^2 - y^2 = 1$. Then R is a transitive relation.

$x^2 - y^2 = 1$ $\quad x^2 - z^2 = (x^2 - y^2) + (y^2 - z^2)$
$y^2 - z^2 = 1$ $\quad = 1 + 1$

T F (3) The relation R on all integers $\mathbb{Z}$ is defined by xRy if and only if $x^2 + y^2 \leq 1$. Then R is a transitive relation.

1 0
0 1

T F (4) There are 4 anagrams of the word MAMA. $\binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \cdot 3}{2} = 2 \cdot 3 = 6$

T F (5) There are infinitely many Fibonacci numbers which are divisible by 3.

T F (6) The number of permutations of $\{1, 2, 3, 4, 5\}$ is smaller than 123. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 20 \cdot 6 = 120$

T F (7) The number of 3-permutations of $\{1, 2, 3, 4, 5, 6\}$ is equal to $\binom{6}{3}$.

T F (8) Sequence $a_n = 10n - n^2$, $n \geq 1$, is increasing. $10-1,\ 20-4,\ 30-9,\ 40-16,\ 50-25,$ $10(n-1) \cdots 110-121$

T F (9) The number of permutations of $\{1, 2, \ldots, n\}$ which have n preceding $n-1$ (not necessarily immediately) is equal to $n!/2$

T F (10) For every $A, B \subset \{1, 2, \ldots, 12\}$ we have $|A \cap B| < |A \cup B|$. $|A \cap B| \leq |A \cup B|$

T F (11) For all $n \geq 1$, we have

$$\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \ldots + \binom{2n}{2n} = 2^{2n-1}.$$

$\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 2^5$

T F (12) The number of grid walks from $(0, 0)$ to $(10, 10)$ going through $(3, 7)$ is equal to $\binom{10}{3}^2$. $\binom{10}{3}$

T F (13) The number of grid walks from $(0, 0)$ to $(10, 10)$ avoiding $(10, 0)$ and $(0, 10)$ is equal to $\frac{1}{2}\binom{20}{10}$.

4S 4I 2P 1M

T F (14) The number of anagrams of MISSISSIPPI which begin with M is greater than the number of anagrams which begin with S. $\frac{10!}{4!4!2!} > \frac{10!}{3!4!2!}$

T F (15) The following parabolas are drawn in the plane:

$$y = x^2 - n^2 x - n^3, \qquad n = 1, \ldots, 12.$$

Then the regions of the plane separated by these parabolas can be colored with two colors in such a way that no two adjacent regions have the same color.

28