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Problem 1. (20 points)

Compute the number of 4-subsets A of $\{1, 2, \ldots, 10\}$ such that:

a) A has no even numbers,

b) A has at least one number ≤ 3,

c) the smallest number in A is divisible by 3.

d) the sum of numbers in A is exactly 11.

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a) $\binom{5}{4}$

2

b) $\binom{10}{4} - \binom{7}{4}$

Why?

2

c) $\binom{7}{3} + \binom{4}{3}$

~~d)~~

$3 \cdot \binom{5}{2} = 30$

Problem 2. (20 points)

Find closed formulas for the following sequences :

a) $a_1 = 1,\ a_{n+1} = a_n \cdot \binom{n+1}{2}$

b) $a_1 = 1,\ a_2 = 1,\ a_{n+1} = a_{n-1} - a_n$ for $n \geq 2$.

Note: you can express a_n in terms of Fibonacci numbers F_n.

a)

n	1	2	3	4	5
a_n	1	1	3	30	

$a_n = ?$

b) $a_{n+1} = a_{n-1} - a_n$

$t^{n+1} = t^{n-1} - t^n$

$t^2 = 1 - t$

$t^2 + t - 1 = 0$

$t = \frac{-1 \pm \sqrt{1+4}}{2}$

$t = \frac{-1 \pm \sqrt{5}}{2}$

$a_1 = 1 = \alpha \frac{-1+\sqrt{5}}{2} + \beta \frac{-1-\sqrt{5}}{2}$

$a_2 = 1 = \alpha \left(\frac{-1+\sqrt{5}}{2}\right)^2 + \beta \left(\frac{-1-\sqrt{5}}{2}\right)^2$

Solve for α & β.

$a_n = ?$

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Problem 3. (15 points)

Let $a_n = 111\cdots1$ (n ones). Suppose a_k is divisible by 97. Use induction to show that $a_{k\cdot n} = 0 \bmod 97$, for all $n \geq 1$.

Base Case: $n=1$, $a_{k\cdot 1} = 0 \bmod 97$

Induction Step: Show that if $a_{k\cdot n} = 0 \bmod 97$, then

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$a_{k(n+1)} = 0 \bmod 97$.

~~Let $A = \{a_k, a_{2k}, a_{3k}, \ldots$~~

Induction step rewritten:

If $1111\ldots11$ ($k\cdot n$ ones) $= 0 \bmod 97$, then $\overbrace{\underbrace{111\ldots1}_{nk \text{ ones}}\underbrace{111\ldots111}_{k \text{ ones}}}^{a_{k(n+1)}} = 0 \bmod 97$

$a_{k\cdot(n+1)} = \underbrace{111\ldots11}_{k \text{ ones}} \cdot 10^{(nk)} + \underbrace{111\ldots11}_{k \text{ ones}} \cdot 10^{(n-1)k} + \ldots = 0 \bmod 97$

k ones: 0 mod 97, induction assumption

k ones: 0 mod 97, induction assumption

$= 0 \bmod 97$

Thus $a_{(n+1)k} = 0 \bmod 97$, all terms in $a_{k\cdot(n+1)} = 0 \bmod 97$.

Problem 4. (15 points)

There are 6 candidates in a student election, to form a 3-student committee. The university printed bulletins with the names of all 6 candidates, and every student must mark exactly 3 of them. The total number of students is 105. Prove that:

a) at least one candidate has 53 votes,

b) at least 6 students cast identical bulletins.

a) Assume all candidates receive less than 53 votes.

Each student casts 3 votes, 105 students, 315 total votes votes.

$\frac{315}{6} > 53$, contradiction.

Thus at least one candidate received more than 53 votes

b). Assume less than 6 cast identical ballots.

Total ballots, $\binom{6}{3}$ options
$= 20$

~~105/20~~

$\frac{105}{20} > 5$, contradiction.

30

$x^2 - x^2 = 0$

Problem 5. (30 points, 2 points each) **TRUE or FALSE?**

24+6 - 26 - 4 = 0m3

Circle correct answer with ink. No explanation required.

T F (1) The number of functions from $\{A, B, C, D, F\}$ to $\{1, 2, 3\}$ is equal to 15.

T F (2) The relation R on integers is defined by $x \to_R y$ if and only if $x^2 - y^2 = 0$. Then R is an equivalence relation.

T F (3) The relation R on integers is defined by $x \to_R y$ if and only if $2x + y = 0$ mod 3. Then R is an equivalence relation.

T F (4) Fibonacci numbers $F_n < (1.99)^n$ for all $n \geq 1$.

T F (5) There are infinitely many Fibonacci numbers which are divisible by 3.

T F (6) The number of permutations of $\{1, 2, 3, 4, 5\}$ is smaller than 101.

T F (7) The number of 3-permutations of $\{1, 2, 3, 4, 5, 6\}$ is equal to $\binom{6}{3}$.

T F (8) Sequence $a_n = 1 + 3/n^2$, $n \geq 1$, is decreasing.

T F (9) The number of permutations of $\{1, 2, \ldots, n\}$ which end with 1 is equal to $n!$

T F (10) For every $A, B \subset \{1, 2, \ldots, 12\}$ we have $|A \cap B| < |A \cup B|$.

T F (11) $a_{n+1} = a_{n-1} + 2a_{n-3} - 4a_{n-5}$ is a linear homogeneous recurrence relation.

T F (12) $a_{n+1} = a_n \cdot a_{n-1}$ is a linear homogeneous recurrence relation.

T F (13) The number of anagrams of MISSISSIPPI which begin with M is greater than the number of anagrams which begin with S.

T F (14) The Inclusion-Exclusion Principle for three sets is the following formula:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$

T F (15) The following parabolas are drawn in the plane:

$$y = x^2 - n^2 x - n^3, \qquad n = 1, \ldots, 12.$$

Then the regions of the plane separated by these parabolas can be colored with two colors in such a way that no two adjacent regions have the same color.