Problem 1. (4 points) Consider an arbitrary natural number $n \geq 2$. Let $A_1, \ldots, A_n$ and C be arbitrary sets. Using mathematical induction, show that

$$\left(\bigcup_{i=1}^{n} A_i\right) \times C = \bigcup_{i=1}^{n} (A_i \times C) \; .$$

Solution: (2 points for basis step; 2 points for inductive step)
Basis step, for $n = 2$. We have that

$$\begin{aligned}
(a, c) \in (A_1 \cup A_2) \times C &\iff a \in A_1 \text{ or } a \in A_2 \text{ and } c \in C \\
&\iff (a, c) \in A_1 \times C \text{ or } (a, c) \in A_2 \times C \\
&\iff (a, c) \in (A_1 \times C) \cup (A_2 \times C)
\end{aligned}$$

And we thus have that $(A_1 \cup A_2) \times C = (A_1 \times C) \cup (A_2 \times C)$.

Inductive step. Assume that $(\bigcup_{i=1}^{n} A_i) \times C = \bigcup_{i=1}^{n} (A_i \times C)$. We have:

$$\begin{aligned}
\left(\bigcup_{i=1}^{n+1} A_i\right) \times C &= \left(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right) \times C \\
&\overset{(1)}{=} \left(\bigcup_{i=1}^{n} A_i\right) \times C \cup (A_{n+1} \times C) \\
&\overset{(2)}{=} \bigcup_{i=1}^{n} (A_i \times C) \cup (A_{n+1} \times C) \\
&= \bigcup_{i=1}^{n+1} (A_i \times C)
\end{aligned}$$

where Equation (1) holds by the basis step, and Equation (2) by the induction assumption.

Problem 2. (4 points) Let R be a relation on a set X.

(a) Explain in words why the statement "R is anti-symmetric" is not the negation of the statement "R is symmetric". Provide examples to illustrate your explanation.

(b) Explain in words why the statement "R is anti-reflexive" is not the negation of the statement "R is reflexive". Provide examples to illustrate your explanation.

Solution:

(a) (1 point for the explanation; 1 point for the counterexample)

Recall that a relation R is symmetric iff for all $x, y \in X$ we have that if xRy then yRx. Thus, a relation R is not symmetric if there exist $x, y \in X$ such that xRy and $y \not R x$. On the other hand, a relation R is anti-symmetric if for all distinct $x, y \in X$ we have that if xRy then $y \not R x$. Thus, if R is anti-symmetric, then there exist $x, y \in X$ such that xRy and $y \not R x$, but the converse is not necessarily true, as the following examples illustrates.

Let $X = \{0, 1, 2\}$ and consider the following relations on X:

$$R = \{(0,1), (0,2), (1,2)\}$$

and

$$R' = \{(0,1), (1,0), (0,2), (1,2), (2,1)\}$$

The relation R is anti-symmetric and thus also not symmetric, as explained above, while relation R' is not symmetric but it is not anti-symmetric, since for instance the two ordered pairs $(1,2)$ and $(2,1)$ violate the condition in the definition of anti-symmetry.

(b) (1 point for the explanation; 1 point for the counterexample)

Recall that a relation R is reflexive iff for all $x \in X$ we have that xRx. Thus, a relation R is not reflexive iff there exists $x \in X$ such that $x \not R x$. On the other hand, a relation R is anti-reflexive iff for all $x \in X$ we have that $x \not R x$. Therefore, if a relation R is anti-reflexive then it is not reflexive, but the converse is not true in general, as the following examples illustrate.

Let $X = \{0, 1, 2\}$ and consider the following relations on X:

$$R = \{(0,1), (0,2), (1,2)\}$$

and

$$R' = \{(0,0),(0,1),(0,2)\}\,.$$

Then R is anti-reflexive, and thus not reflexive, as explained above. On the other hand, R' is not reflexive, but it is not anti-reflexive, because the ordered pair $(0,0)$ violates the condition in the definition of anti-reflexivity.

Problem 3. (3 points) Let n be a positive natural number. Let $X = \{i \in \mathbb{N} \colon 1 \leq i \leq n\}$. Denote by $\mathcal{P}(X)$ the power set of X, and let $\mathcal{P}^\star(X) := \mathcal{P}(X) \setminus \emptyset$ denote the set of subsets of X that are not empty. Consider the function

$$f \colon \mathcal{P}^\star(X) \to X$$

which sends each non-empty subset of X to its least element. For instance, $f(\{1,3\}) = 1$. For which values of n is f injective, surjective, or bijective? Carefully motivate your arguments.

Solution:

(1 point for showing that f is a bijection for $n = 1$)

If $n = 1$, then $X = \{1\}$ and the only non-empty subset of X is $\{1\}$. Because there are no two distinct elements in $P^*(X)$, the function f is injective. Moreover, $f(\{1\}) = 1$ so 1 is in the image of f. As 1 is the only element in the codomain, this shows f is also surjective. Hence f is a bijection when $n = 1$.

(For $n > 1$: 1 point for showing that f is not an injection; 1 point for showing that f is a surjection)

Now suppose $n > 1$. Thus $n \geq 2$ and so $\{1, 2\}$ is a non-empty subset of X. Note that

$$f(\{1\}) = 1 = f(\{1, 2\}),$$

so f is not injective. If $i \in X$, we know $\{i\}$ is a non-empty subset of X and $f(\{i\}) = i$ so i is in the image of f. This shows f is surjective.
In sum, we have shown that f is surjective for all $n \in \mathbb{N}$ and injective (and hence bijective) if and only if $n = 1$.

Problem 4. (2 points) A teacher wants to arrange their 11 students in a single line. There are two students called Averie and Charlie in this class. How many ways are there for the students to line up so that Averie is first in line or Charlie is last?

Solution:
(1 point for computing the number of ways to order students with Averie first / Charlie last)

First, we calculate the number of ways there are to line up the students with Averie first in line. To order students in this way, we first place Averie in front, and select an ordering of the remaining 10 students. There are 10! many such orderings.
Likewise, to select a way of lining up the students with Charlie last, we place Charlie last and select an ordering of the remaining 10 students. There are 10! many such orderings.

(1 point for calculating the number of ways to put Averie first and Charlie last, and using the inclusion-exclusion principle)

Now we calculate how many orderings there are with Averie first and Charlie last. For this, we place Averie first and Charlie last and then select an ordering of the remaining 9 students. There are 9! many such orderings.
By the inclusion-exclusion principle, that results in $2 \times 10! - 9!$ many orderings.