Math 61

Midterm 1

Fall, 2016

Name: Key

SID:

Section:

There are 5 questions. Write clearly, show all of your work, and justify all of your answers. No calculators are allowed.

1	
2	
3	
4	
5	
Total	

1. (15 pts)

(a) (5 pts) Write the definition of what it means for a relation R on a set X to be an equivalence relation.

An equivalence relation is a reflexive, symmetric, and transitive relation.

(b) (6 pts) Let E be the relation on the set of all positive real numbers $\mathbb{R}^+$ where $x \; E \; y$ if $x/y = 2^n$ for some integer $n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. For example, $(2/3) \; E \; (8/3)$, since $\frac{2/3}{8/3} = \frac{1}{4} = 2^{-2}$. Show that E is an equivalence relation.

For every $x \in \mathbb{R}^+$, $x/x = 1 = 2^0$ and so $x E x$. So E is reflexive.
For every $x, y \in \mathbb{R}^+$, if $x E y$, then $x/y = 2^n$ for some integer n. Hence $y/x = 2^{-n}$, so $y E x$ since $-n$ is an integer. So E is symmetric.
For every $x, y, z \in \mathbb{R}^+$, if $x E y$ and $y E z$, then $x/y = 2^n$ for some integer n and $y/z = 2^m$ for some integer m. Hence $x/z = \frac{x/y}{z/y} = \frac{2^n}{2^{-m}} = 2^{n+m}$ and so $x E z$ since $n+m$ is an integer. So E is transitive.

(c) (4 pts) What is the equivalence class of 1 with respect to the relation E?

$[1]_E = \{x \in \mathbb{R}^+ : x E 1\} = \{x \in \mathbb{R}^+ : x/1 = 2^n \text{ for some integer } n\}$

$= \{\ldots, 2^{-2}, 2^{-1}, 2^0, 2^1, 2^2, \ldots\}$

all integer powers of 2.

2. (15 pts)

Prove the following by induction for every $n \geq 1$. For all finite sets X and Y with $|X| = |Y| = n$, if f is a function from X to Y that is one-to-one, then f is onto.

(a) (3 pts) State and prove the base case.

For all finite sets X and Y with $|X| = |Y| = 1$, if f is a function from X to Y, that is one-to-one, then f is onto.
This is true since if $|X| = |Y| = 1$, both X and Y have only one element. So if we call x the unique elt of X and y the unique elt of Y, we must have $f(x) = y$, so f is onto.

(b) (9 pts) Prove the inductive step.

Suppose that for all finite sets X and Y with $|X| = |Y| = k$, if f is a 1-1 function from X to Y, then f is onto. Now let X and Y be finite sets where $|X| = |Y| = k+1$, and $f: X \to Y$ be 1-1. Let a be an element of X, let $X' = X - \{a\}$, let $b = f(a)$, and let $Y' = Y - \{b\}$. Let f' be the function on X' where $f'(x) = f(x)$ for all $x \in X'$.

- For all $x \in X'$, since $x \neq a$ $f'(x) = f(x) \neq f(a) = b$, so $f'(x) \in Y'$.
  So f' is a function from X' to Y'. Note $|X'| = |Y'| = k$
- For all $x_1, x_2 \in X'$ that are distinct, $f'(x_1) = f(x_1) \neq f(x_2) = f'(x_2)$ since f is 1-1.
  So f' is 1-1. So our induction hypothesis applies to f' from X' to Y'.

Now suppose $y \in Y$. If $y = b$, then $f(a) = y$. If $y \neq b$, then $y \in Y'$ so by our IH, $y = f'(x) = f(x)$ for some $x \in X' \subseteq X$. So f is onto.

(c) (3 pts) Either prove the following or give a counterexample: For every set X, if f is a one-to-one function from X to X, then f is onto.

This is false. For example, let $f: \{0,1,2,\ldots\} \to \{0,1,2,\ldots\}$ be the function $f(x) = x+1$. f is 1-1, since if $f(x_1) = f(x_2)$, then $x_1 + 1 = x_2 + 1$ so $x_1 = x_2$. But f is not onto since there is no $x \geq 0$ such that $f(x) = x+1 = 0$.

3. (10 pts) **In the following problem, for full credit, state any counting rules or principles that you use. You do not need to simplify your answers (for example, they may contain factorials). However, your final answer may not include the functions $P(n, r)$ or $C(n, r)$.**

(a) (5 pts) How many solutions are there to $x_1 + x_2 + \ldots + x_9 = 20$, where $x_1, x_2, \ldots, x_9$ are integers greater than or equal to 0?

Use stars and bars with 20 stars and 8 bars to split up the indistinguishable stars into 9 variables.

$$C(20+8, 8) = \frac{28!}{20!\,8!}.$$

(b) (5 pts) How many solutions are there to $x_1 + x_2 + \ldots + x_9 = 20$, where $x_1, x_2, \ldots, x_9$ are integers greater than or equal to 0, where at least one of the variables $x_1, \ldots, x_9$ is equal to 0 or exactly one of the variables $x_1, \ldots, x_9$ is equal to 1?

Let A be the set of solutions where $x_i > 0$ for all i. If we allocate 1 star to each variable before we begin, there are 11 stars left to distribute among 9 variables so by stars and bars, $|A| = C(11+8, 8) = C(19, 8)$

Let B be the set of solutions where $x_i > 0$ for all i and $x_i = 1$ for exactly one i. There are 9 ways to choose this one value i where $x_i = 1$ and then if we allocate 2 stars to each of the other 8 variables, there are $20 - 1 - 2 \cdot 8 = 3$ stars left to allocate among 8 variables. So by stars and bars, $|B| = C(3+7, 7) = C(10, 7)$. Note $B \subseteq A$

$A - B$ is the set of solutions where none of the variables are equal to 0 and it is not the case that exactly one variable is equal to 1. $|A - B| = C(19, 8) - C(10, 7)$ by the above. The set of solutions we wish to count is everything that remains.

$$C(28, 8) - (C(19, 8) - C(10, 7)) = \frac{28!}{20!\,8!} - \frac{19!}{11!\,8!} + \frac{17!}{3!\,7!}$$

4. (15 pts) Consider a standard deck of 52 playing cards, where each card is one of four different suits $\diamondsuit, \heartsuit, \clubsuit, \spadesuit$ and one of 13 different ranks: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, and all combinations of suits and ranks are possible.

A hand means 5 different cards where order does not matter.

For full credit, state any counting rules or principles that you use. You do not need to simplify your answers (for example, they may contain factorials). However, your final answer may not include the functions $P(n,r)$ or $C(n,r)$.

(a) (5 pts) How many hands contain only cards whose rank is J or Q or K?

There are $3 \cdot 4 = 12$ cards whose rank is J or Q or K.
There are $C(12,5)$ ways to choose 5 of them. (Combinations)

$$C(12,5) = \frac{12!}{7!5!}$$

(b) (5 pts) How many hands are "three of a kind" (contain three cards of one rank, and the remaining cards have two other ranks)?

There are 13 ways to choose the rank for the "three-of-a-kind", $C(4,3)$ ways to choose 3 suits for the cards of this rank, $C(12,2)$ ways to choose two other ranks, and then $C(4,1)$ ways to choose a suit for each such rank. So by the multiplication principle,

$$13 \cdot C(4,3) \cdot C(12,2) \cdot C(4,1) \cdot C(4,1) = 13 \cdot \frac{4!}{1!3!} \cdot \frac{12!}{2!2!} \cdot 4 \cdot 4$$

(c) (5 pts) How many hands are "two pair" (contain two cards of the same rank, two cards of another rank, and one card of a third rank)?

There are $C(13,2)$ ways to choose two ranks to be pairs, then $C(4,2)$ ways to choose suits for each pair, there are $C(11,1)$ ways to choose a last rank and $C(4,1)$ ways to choose a suit for this last card. So by the multiplication principle,

$$C(13,2) \cdot C(4,2) \cdot C(4,2) \cdot C(11,1) \cdot C(4,1) = \frac{13!}{11!2!} \frac{4!}{2!2!} \frac{4!}{2!2!} \cdot 11 \cdot 4$$

5. (20 pts) Circle whether the following are True or False. **You do not need to justify these answers.**

T / F: If A and B are finite sets and $|A \cup B| = |B|$, then $A \subseteq B$.

T / F: If X is a finite set with $|X| = k$, then there are k^n many strings over X of length n.

T / F: If f is a bijection from X to Y, then f^{-1} is a bijection from Y to X.

T / F: There are $\frac{(n+m)!}{n!m!}$ ways to divide n many identical balls into m many distinct boxes.

T / F: Consider the set S of strings of length 10 containing exactly four a's, three b's, and three c's. There are more strings in S ending with a than there are strings in S ending with b.

T / F: Suppose f is a function from X to Y. Then f is one-to-one if for every $x \in X$ there is a unique $y \in Y$ so that $f(x) = y$.

T / F: Suppose R is a relation on a set X and R is transitive and symmetric. Then for all $x, y \in X$, if $(x, y) \in R$, then $(x, x) \in R$.

T / F: The relation $\{(1,2), (3,1), (2,3), (1,1), (2,2), (3,3)\}$ on the set $\{1,2,3\}$ is transitive.

T / F: The number of onto functions from $\{1,2,3,4,5\}$ to $\{1,2\}$ is equal to $\frac{5!}{3!2!}2^3$.

T / F: If $n \geq 3$, then there are $(n-1)!/2$ ways to seat n different people around a circular table where two seatings are considered identical if each person has the same (unordered) set of two neighbors.