Name: ____

UCLA ID Number: ______________________________________

Section letter: __________

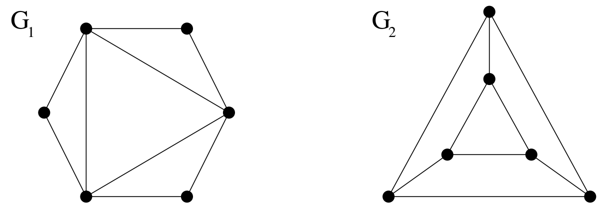
Math 61 : Discrete Structures
Midterm 2
Instructor: Ciprian Manolescu

You have 50 minutes.

No books, notes or calculators are allowed.
Do not use your own scratch paper.

1. (10 points) **True or False:** Circle the right answers. You do NOT need to justify your answers.

Consider the following graphs:



The incidence matrices of G_1 and G_2 have the same number of rows. **T**

The incidence matrices of G_1 and G_2 have the same number of columns. **T**

G_1 and G_2 are isomorphic. **F**

G_1 and G_2 both have Hamiltonian cycles. **T**

G_1 admits an Euler cycle. **T**

G_2 admits an Euler cycle. **F**

G_1 admits an Euler path between different vertices. **F**

G_2 admits an Euler path between different vertices. **F**

Unrelated to the picture above:

There exists a graph with six vertices, of degrees $4, 4, 4, 4, 4, 5$. **F**

There exists a graph with six vertices, of degrees $4, 4, 4, 4, 5, 5$. **T**

2. (6 points) Write down the answer to the following questions. *You do NOT need to justify your answers.*

If a graph has 100 vertices, all of degree 3, then how many edges does it have?

The sum of the degrees $= 100 \cdot 3 = 300 =$ twice the number of edges, so it has 150 edges

Let d_n be the number of ways one can fill a $2 \times n$ rectangle with 2×1 dominoes. Write down a recurrence relation for d_n.

If we place the first domino horizontally we are left with the problem of filling a $2 \times (n-1)$ rectangle, which can be done in d_{n-1} ways. Otherwise, we must place the first two dominoes vertically side by side, and then we are left with the problem of filling a $2 \times (n-2)$ rectangle, which can be done in d_{n-2} ways. Overall, we find

$$d_n = d_{n-1} + d_{n-2}$$

3. (8 points) Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2}$$

with the initial conditions $a_0 = 2, a_1 = 2$. *Show all your work.*

Solving the equation $t^2 = 2t + 1$ we find the roots $1 + \sqrt{2}$ and $1 - \sqrt{2}$, so the solution is of the form

$$a_n = A(1 + \sqrt{2})^n + B(1 - \sqrt{2})^n.$$

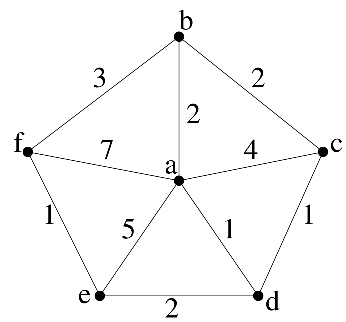
From $a_0 = 2$ we find $A + B = 2$ and from $a_1 = 2$ we find $A(1 + \sqrt{2}) + B(1 - \sqrt{2}) = 2$. From here we get $A = B = 1$, and therefore

$$a_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n.$$

4. (8 points) Ten objects are placed on a square of side length 3 feet. Prove that there exist two objects at distance at most $\sqrt{2}$ feet from each other. *Show all your work.*

Split the square into nine (3×3) small squares of side length 1 foot. By the pigeonhole principle, there are two objects in the same small square. The biggest possible distance between them is the diagonal, which is of length $\sqrt{2}$.

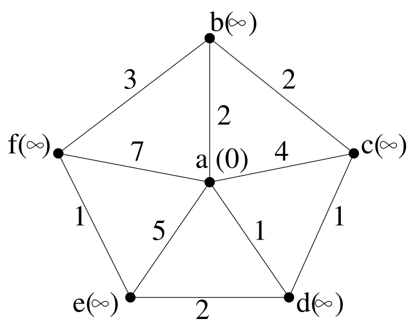
5. (8 points) Consider the following weighted graph:



Suppose you want to use Dijkstra's algorithm to find the length of the shortest path from a to f. Write down the initialization and the first two iterations of the algorithm. (You do NOT have to complete the whole algorithm.) At each step, write down if f is in T, what is the current node, what is the new set of unvisited vertices T, then which nodes change labels and how.

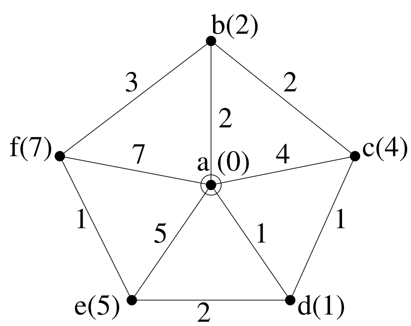
Also, please circle the visited labels at each step (the ones not in T), and write down the label at each vertex in paranthesis.

Initialization:



T={a,b,c,d,e,f}

First iteration:



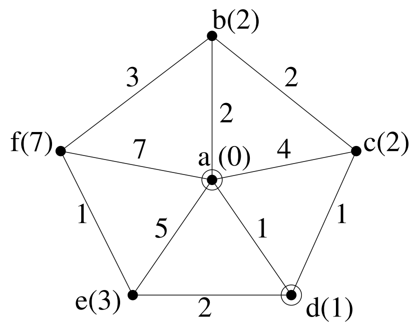
Is f in T? Yes

current node: a

T={b,c,d,e,f}

changes in labels: b: min(∞,2)=2, c: min(∞,4)=4, d: min(∞,1)=1
e: min(∞,5)=5, f: min(∞ ,7)=7

Second iteration:



Is f in T? Yes

current node: d

T={b,c,e,f}

changes in labels: c: min(4,1+1)=2, e: min(5,1+2)=3

Do not write on this page.

1		out of 10 points
2		out of 6 points
3		out of 8 points
4		out of 8 points
5		out of 8 points
Total		out of 40 points