Name: ____

UCLA ID Number: ______________________________________

Section letter: __________

Math 61 : Discrete Structures
Solutions to Midterm 1
Instructor: Ciprian Manolescu

You have 50 minutes.

No books, notes or calculators are allowed.
Do not use your own scratch paper.

1. (8 points) **Circle the right answer. You do NOT need to justify your answers.**

(a) (1 point each) **True / False:** Consider the following relation on the set $\mathbb{Z}$ of integers:

$$xRy \iff x = y = 0.$$

Then:

R is reflexive **T** / $\boxed{\textbf{F}}$

R is symmetric $\boxed{\textbf{T}}$ / **F**

R is antisymmetric $\boxed{\textbf{T}}$ / **F**

R is transitive $\boxed{\textbf{T}}$ / **F**

(b)(2 points each) **Multiple choice:** Consider the sequences a and b defined by

$$a_n = (-1)^n, \quad b_n = (-1)^{n+1}$$

for $n \geq 1$. Then:

(A) a is a subsequence of b, but b is not a subsequence of a

(B) b is a subsequence of a, but a is not a subsequence of b

$\boxed{(C)}$ a and b are subsequences of each other

(D) a is not a subsequence of b, and b is not a subsequence of a

In how many ways can we distribute 50 identical cookies to 7 distinct people, such that every person gets at least 2 cookies?

$$(A)\ \frac{36!}{7! \cdot 29!} \qquad (B)\ \frac{36!}{6! \cdot 30!} \qquad (C)\ \frac{42!}{7! \cdot 35!} \qquad \boxed{(D)}\ \frac{42!}{6! \cdot 36!}.$$

2. (2 points each) Write down the answer to each question. **You do NOT need to justify your answers.** Also, you do not need to simplify expressions such as 2^6, $6!$, $C(6,3)$, etc.

However, the answer should be in closed form, not as a summation.

(a) Consider the relation R from $X = \{1,2,3\}$ to $Y = \{2,3,4\}$ given by

$$xRy \iff x < y.$$

Write down the matrix of R.

$$\begin{array}{c c} & \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

(b) Consider a string of length n, made of n distinct letters. How many substrings does it have?

If the substring is nonempty, we can choose it by placing a bar at its beginning and a bar at its end, as in $a_1|a_2a_3a_4|a_5a_6$ giving the substring $a_2a_3a_4$ or $|a_1a_2a_3a_4a_5|a_6$ giving $a_1a_2a_3a_4a_5$. There are $n+1$ possible positions for the bars, of which we have to choose two, in $C(n+1,2)$ ways. To this we add the null string λ, for a total of

$$C(n+1,2)+1 = \frac{n(n+1)}{2}+1.$$

(c) Consider a sequence of length n, made of n distinct letters. How many subsequences does it have? For each letter, we have to decide whether it should go in the subsequence.

There are n steps and 2 choices at each step, so by the multiplication principle we get 2^n subsequences.

(d) Calculate the sum

$$\sum_{n=10}^{90} 2 \cdot (-3)^n.$$

By substituting $k = n - 10$ we get

$$\begin{aligned}
\sum_{n=10}^{90} 2 \cdot (-3)^n &= \sum_{k=0}^{80} 2 \cdot (-3)^{k+10} \\
&= \sum_{k=0}^{80} 2 \cdot (-3)^{10} \cdot (-3)^k \\
&= 2 \cdot (-3)^{10} \sum_{k=0}^{80} (-3)^k \\
&= 2 \cdot (-3)^{10} \frac{(-3)^{81} - 1}{(-3) - 1} \\
&= 2 \cdot 3^{10} \frac{3^{81} + 1}{4} \\
&= \frac{3^{91} + 3^{10}}{2}.
\end{aligned}$$

(e) Consider the strings of length 10 made from the 26 letters of the English alphabet, such that they contain exactly three A's, exactly three B's, and no other letters are repeated. (An example of such a string is $BQWABAAXBM$.) How many such strings are there?

We can choose the positions of the A's in $C(10, 3)$ ways, then the positions of the B's in $C(7, 3)$ ways, and finally the rest in $P(24, 4)$ ways. The answer is

$$C(10, 3) \cdot C(7, 3) \cdot P(24, 4) = \frac{10!}{3! \cdot 7!} \cdot \frac{7!}{3! \cdot 4!} \cdot \frac{24!}{20!}.$$

(f) Consider the strings of length 10 made from the 26 letters of the English alphabet, such that they contain at least one A and at least one B. (An example of such a string is $RTTADBLBBR$.) How many such strings are there?

Let X_1 be the set of strings that do not contain A, and X_2 the set of strings that do not contain B. Then

$$|X_1| = 25^{10}, \quad |X_2| = 25^{10}$$

Also, $X_1 \cap X_2$ is the set of strings that do not contain either A or B (so there are 24 letters left to choose from). We have

$$|X_1 \cap X_2| = 24^{10}.$$

By inclusion-exclusion,

$$|X_1 \cup X_2| = |X_1| + |X_2| - |X_1 \cap X_2| = 2 \cdot 25^{10} - 24^{10}.$$

Note that $X_1 \cup X_2$ consists of those strings that are missing either A or B or both. What we are interested in is the complement of $X_1 \cup X_2$. Since there are 26^{10} strings total, the answer is

$$26^{10} - 2 \cdot 25^{10} + 24^{10}.$$

3. (a) (1 point) Write the definition of a one-to-one (injective) function.

$f : X \to Y$ is injective if for every $x_1, x_2 \in X$, if $f(x_1) = f(x_1)$ then $x_1 = x_2$.

Next, answer the following questions, fully justifying your answers. (If an answer is YES, explain why. If an answer is NO, give a counterexample.)

Suppose we have functions $g : X \to Y$ and $f : Y \to Z$, with composition $f \circ g : X \to Z$.

(a) (3 points) If $f \circ g$ is one-to-one, does f have to be one-to-one?

No. Counterexample: Take $X = \{1\}$, $Y = \{1, 2\}$, $Z = \{1\}$ and $g(1) = 1$, $f(1) = f(2) = 1$. Then f is not one-to-one but $f \circ g$ is the identity on $\{1\}$, so it's one-to-one.

(b) (3 points) If f is one-to-one, does $f \circ g$ have to be one-to-one?

No. Counterexample: Take $X = \{1, 2\}$, $Y = \{1\}$, $Z = \{1\}$ and $g(1) = g(2) = 1$, $f(1) = 1$. Then $f \circ g$ is not one-to-one but f is the identity on $\{1\}$, so it's one-to-one.

(c) (3 points) If f and g are one-to-one, does $f \circ g$ have to be one-to-one?

Yes. If $(f \circ g)(x_1) = (f \circ g)(x_2)$, we have $f(g(x_1)) = f(g(x_2))$. Set $y_1 = g(x_1)$, $y_2 = g(x_2)$. We get $f(y_1) = f(y_2)$. Since f is one-to-one, we get $y_1 = y_2$, so $g(x_1) = g(x_2)$. Since g is one-to-one, we get $x_1 = x_2$. This means that $f \circ g$ is one-to-one.

4. (10 points) Prove by induction on n that:

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$$

for any integer $n \geq 1$.

The base case is $n = 1$, when

$$1^2 = \frac{(-1)^2 1 \cdot 2}{2}$$

is true.

Suppose the equality is true for n. For $n+1$, we have

$$\begin{aligned}
1^2 - 2^2 + 3^2 - \cdots + (-1)^{n+1} n^2 + (-1)^{n+2}(n+1)^2 &= \frac{(-1)^{n+1} n(n+1)}{2} + (-1)^{n+2}(n+1)^2 \\
&= (-1)^{n+2} \cdot (-1) \frac{n(n+1)}{2} + (-1)^{n+2}(n+1)^2 \\
&= (-1)^{n+2} \cdot \left(-\frac{n(n+1)}{2} + (n+1)^2 \right) \\
&= (-1)^{n+2}(n+1) \cdot \left(-\frac{n}{2} + (n+1) \right) \\
&= (-1)^{n+2}(n+1) \cdot \left(\frac{n}{2} + 1 \right) \\
&= \frac{(-1)^{n+2}(n+1)(n+2)}{2}
\end{aligned}$$

By induction, the statement is true for all $n \geq 1$.