1. (2 points each) **Multiple choice:** Circle the right answer. You do NOT need to justify your answers.

If α is a string of length two, what is the number of substrings of α?

(A) 2; (B) 3; (C) 4; (D) 5; $\boxed{(E)}$ It depends on the string.

Explanation: The string aa has 3 substrings, but ab has 4.

In how many ways can we distribute 12 distinct books to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; $\boxed{(B)}$ 5^{12}; (C) $C(12,5)$; (D) $C(16,4)$; (E) $C(16,11)$.

Explanation: We have to choose from 5 people 12 times. We use the multiplication principle to count.

In how many ways can we distribute 12 (identical) copies of the same book to 5 people: Alice, Bob, Casey, David and Emily?

(A) 12^5; (B) 5^{12}; (C) $C(12,5)$; $\boxed{(D)}$ $C(16,4)$; (E) $C(16,11)$.

Explanation: We choose 12 people out of 5 with repetitions. The number of such combinations with repetitions is $C(12+5-1, 5-1)$.

Let $X = \{1,2,3,4\}$. If $f, g : X \to X$ are two functions such that $f \circ g$ is bijective, then:

(A) f has to be bijective, but g does not have to be bijective;

(B) g has to be bijective, but f does not have to be bijective;

$\boxed{(C)}$ Both f and g have to be bijective;

(D) Neither f nor g have to be bijective.

Explanation: Since $f \circ g$ is surjective, we have that f has to be surjective (it must hit all the values). A surjective function between sets with the same number of elements is bijective. Hence f is bjiective.

Since $f \circ g$ is injective, we have that f has to be injective. An injective function between sets with the same number of elements is bijective. Hence f is bjiective.

2. Write down the answer to each question. *You do NOT need to justify your answers.* Also, you do not need to simplify expressions such as 2^6, $6!$, $C(6,3)$, etc.

(a) Consider the set $X = \{1, 2, \ldots, 10\}$.

(2 points) How many of the relations on X are NOT reflexive?

$2^{100} - 2^{90}$

(2 points) How many of the relations on X are symmetric?

$2^{10\cdot 11/2} = 2^{55}$

(2 points) How many of the relations on X are symmetric OR reflexive (i.e. either symmetric, or reflexive, or both)?

$2^{55} + 2^{90} - 2^{45}$

(2 points) How many of the relations on X are symmetric but NOT reflexive?

$2^{55} - 2^{45}$

(2 points) How many of the relations on X are both symmetric AND antisymmetric?

2^{10} (for the matrix of the relation, we need to have 0's outside the diagonal, and we are free to choose 0 or 1's on the diagonal)

(b) (2 points) How many distinct strings can be obtained from the string $AAABBCCCD$ by permuting (re-ordering) its letters?

$$\frac{9!}{3! \cdot 2! \cdot 3! \cdot 1!}$$

3. Let $\mathbb{Z}$ be the set of all integers, and $\mathcal{P}(\mathbb{Z})$ the power set of $\mathbb{Z}$ (consisting of all subsets of $\mathbb{Z}$). Consider the following relation R on $\mathcal{P}(\mathbb{Z})$:

$$(A, B) \in R \iff A \cap B \neq \emptyset.$$

Answer the following questions, fully justifying your answers. (If an answer is YES, explain why. If an answer is NO, give a counterexample.)

(a) (2 points) Is R reflexive?

No: $(\emptyset, \emptyset) \notin R$

(b) (2 points) Is R symmetric?

Yes: $A \cap B \neq \emptyset \iff B \cap A \neq \emptyset$

(c) (3 points) Is R transitive?

No: Let $A = \{1, 2\}, B = \{2, 3\}, C = \{3, 4\}$. Then $(A, B) \in R$ and $(B, C) \in R$ but $(A, C) \notin R$.

(d) (3 points) Prove that $(A, B) \in R \circ R$ whenever A and B are nonempty.

Let $C = A \cup B$. Then $(A, C) \in R$ and $(C, B) \in R$, hence $(A, B) \in R \circ R$.

4. (10 points) Prove by induction on n that:

$$3^n \leq (n+1)!$$

for any integer $n \geq 4$.

Base case $n = 4$: $3^4 = 81 \leq 5! = 120$ is true.
Inductive step: Suppose $3^n \leq (n+1)!$ and we want to show $3^{n+1} \leq (n+2)!$.
Multiplying $3^n \leq (n+1)!$ by 3 we get $3^{n+1} \leq 3 \cdot (n+1)!$. It now suffices to show that

$$3 \cdot (n+1)! \leq (n+2)! = (n+1)! \cdot (n+2).$$

Dividing both sides by $(n+1)!$, this is equivalent to $3 \leq n+2$, or $1 \leq n$, which is true.