Midterm 2 Solutions

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Problem 1

Determine if the following are true or false.

Part a

False. $K_{61,2019}$ has vertices of odd degree 61 and so cannot have an Euler cycle.

Part b

False. The sum of the degrees of all vertices is always even! This is because $\sum_{v \in V} \delta(v) = 2|E|$ is even. So we cannot have 9 vertices each of degree 3, as the sum of degrees would then be 27, which is odd.

Part c

False. Vertices 2 and 4 have no edges to any of vertices $1, 3, 5$.

Part d

True. Color the vertices with two colors and this becomes clear.

Part e

True. Let a map to 5, b map to 1, c map to 3, d to 6, e to 2, and f to 4.

Problem 2

Part a

With the ansatz $a_n = r^n$, we see

$$a_n = 4a_{n-1} + 5a_{n-2}$$

gives

$$r^n = 4r^{n-1} + 5r^{n-2}$$

and hence

$$r^2 = 4r + 5$$

Solving for r, we see

$$r^2 - 4r - 5 = 0$$

$$(r-5)(r+1) = 0$$

so $r = 5$ or $r = -1$. Thus the general solution is

$$a_n = c_1 \cdot 5^n + c_2 \cdot (-1)^n$$

Part b

With our initial conditions we see

$$5 = a_0 = c_1 \cdot 5^0 + c_2 \cdot (-1)^0 = c_1 + c2$$

$$7 = a_1 = c_1 \cdot 5^1 + c_2 \cdot (-1)^1 = 5c_1 - c_2$$

Adding these equations we see

$$12 = 6c_1$$

$$c_1 = 2$$

$$c_2 = 5 - c_1 = 3$$

so

$$a_n = 2 \cdot 5^n + 3 \cdot (-1)^n$$

Part c

We have

$$b_n = b_{n-1}^4 b_{n-2}^5$$

Taking log base two of both sides, we see

$$\log_2(b_n) = 4\log_2(b_{n-1}) + 5\log_2(b_{n-2})$$

Letting $a_n = \log_2(b_n)$, we see

$$a_n = 4a_{n-1} + 5a_{n-2}$$

which by part a gives us the general solution

$$a_n = c_1 \cdot 5^n + c_2 \cdot (-1)^n$$

Since $a_n = \log_2(b_n)$, and $b_0 = 1, b_1 = 16$, the initial conditions translate to $a_0 = \log_2(1) = 0$ and $a_1 = \log_2(16) = 4$.

Plugging these into our general solution we see

$$0 = a_0 = c_1 + c_2$$

$$4 = a_1 = 5c_1 - c_2$$

Adding these equations gives

$$4 = 6c_1$$

so $c_1 = 4/6 = 2/3$, and $c_2 = -c_1 = -2/3$.

So we get the solution

$$a_n = \frac{2}{3} \cdot 5^n - \frac{2}{3} \cdot (-1)^n$$

Translating to a solution for b_n, we get

$$b_n = 2^{a_n} = 2^{\left(\frac{2}{3} \cdot 5^n - \frac{2}{3} \cdot (-1)^n\right)}$$

Part d

To solve

$$c_n = 4c_{n-1} + 5c_{n-2} + 16$$

we first compute the homogeneous solution, i.e. the general solution to

$$c_n^H = 4c_{n-1}^H + 5c_{n-2}^H$$

By part a, the general solution to this is

$$c_n^H = d_1 \cdot 5^n + d_2 \cdot (-1)^n$$

Next, we compute a particular solution. Since the extra forcing term is constant, and $r = 1$ is not a solution to the characteristic polynomial in part a, we may find a particular solution of the form $c_n = C$, where C is just a constant. Plugging this into our recurrence relation, we see

$$C = 4C + 5C + 16$$

so $-8C = 16$, so $C = -2$. Thus, a particular solution is $c_n = -2$ for all n.

Finally, our general solution to this recurrence relation is the sum of the general homogenous and particular solution, which gives

$$c_n = d_1 \cdot 5^n + d_2 \cdot (-1)^n - 2$$

Finally we are ready for our initial conditions. Since $c_0 = 0$ and $c_1 = 2$, we see

$$0 = c_0 = d_1 + d_2 - 2$$

$$2 = c_1 = 5d_1 - d_2 - 2$$

Adding these equations we see

$$2 = 6d_1 - 4$$

so $d_1 = 1$, and $d_2 = 2 - d_1 = 1$. Hence,

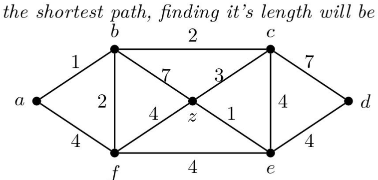
$$c_n = 5^n + (-1)^n - 2$$

Problem 3

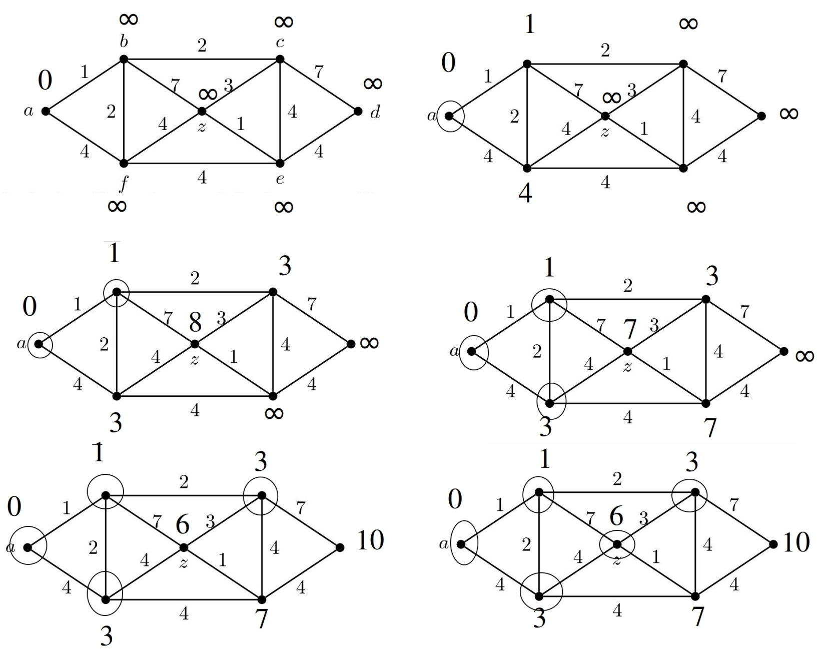
See the attached image for the correct steps.

Name:______________________________

3. [20 pts] Use Dijkstra's algorithm to find the length of the shortest path (i.e. the path for which the sum of the labels is as small as possible) between a and z in the weighted graph below. *You do not need to find the shortest path, finding it's length will be sufficient.*



Show each step of Dijkstra's algorithm. A correct final answer with no work shown will not be sufficient for full credit. Use the blank graphs below for your answer. If you make a mistake, clearly cross it out and continue using the next blank graph. There are additional blank graphs on the back of this page.



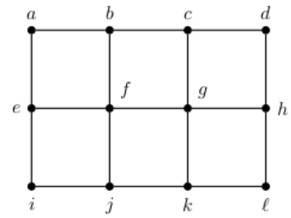
Answer: 6

Check this box if you used any graphs from the back of the page: ☐

5

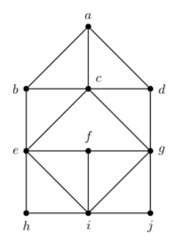
Problem 4

Part a



We have a Hamiltonian cycle $a - b - c - d - h - l - k - g - f - j - i - e - a$.

Part b



Suppose this graph $G = (V, E)$ had a Hamiltonian cycle $H = (V, E')$ with $E' \subset E$. Recall that since H is a hamiltonian cycle, H must be connected, have $|E'| = |V|$ and $\delta_H(v) = 2$ for all $v \in V$. We will make use of the last property.

First, since $\delta_H(h) = 2 = \delta_G(h)$, we must have both $(h, i) \in E'$ and $(h, e) \in E'$. (In words, since h has degree precisely two, both edges incident to it must be present in any Hamiltonian cycle).

Similarly, since $\delta_H(j) = 2 = \delta_G(j)$, we must have $(j, i) \in E'$ and $(j, g) \in E'$.

Next, since $(i, h) \in E'$ and $(i, j) \in E'$, and $\delta_H(i) = 2$, we must have $(i, e), (i, f), (i, g) \notin E'$. (In words, since i already has two incident edges in the Hamiltonian cycle, the remaining edges adjacent to it in G must not be in the Hamiltonian cycle).

Next, since $\delta_H(f) = 2$, $\delta_G(f) = 3$, and $(f, i) \notin E'$ by the above, we must have $(f, e) \in E'$ and $(g, e) \in E'$. (H must contain precisely two of the three incident edges to f, and we already ruled one edge out).

Next, since $\delta_H(e) = 2$ and $(e, h) \in E'$ and $(e, f) \in E'$, we must have $(e, i) \notin E'$, $(e, c) \notin E'$, and $(e, b) \notin E'$.

Similarly, $\delta_H(g) = 2$ and $(g, j), (g, f) \in E'$, so that $(g, i), (g, c), (g, d) \notin E'$.

Now we are ready to contradict! By assumption, H is a hamiltonian cycle and therefore connected. However, H is a subgraph of G with $(e, b), (e, c), (g, c), (g, d) \notin E'$. Then there is no path from any of the bottom vertices e, f, g, h, i, j to any of the top vertices a, b, c, d. This is clear from the corresponding picture deleting those 4 edges from G. More formally, in G, the only edges going from $\{e, f, g, h, i, j\}$ to $\{a, b, c, d\}$ are $(e, b), (e, c), (g, c), (g, d)$, and none of these are in H, so H is disconnected.

By contradiction, no such Hamiltonian cycle H can exist, and G does not have a Hamiltonian cycle.

Problem 5

Part a

Method 1: Partition G into its connected components. Write $V = V_1 \cup V_2 \cup ... \cup V_k$ for the corresponding partition of vertices into k (nonempty) connected components. Since this is a partition, $V_i \cap V_j = \emptyset$ for $i \neq j$.

Let $v \in V_i$ be arbitrary. Since $\delta_G(v) \geq 5$ and G is simple, we see v has at least five distinct neighbors (it has no edges to itself nor multiple edges to the same vertex). These neighbors have a path to v and thus are also in the same connected component V_i. Thus, $|V_i| \geq 6$ (it contains v and its 5 or more neighbors).

This holds for any i, so that $|V_i| \geq 6$ for all $i = 1, ..., k$. Meanwhile,

$$10 = |V| = |V_1| + |V_2| + ... + |V_k| = \sum_{i=1}^{k} |V_i| \geq \sum_{i=1}^{k} 6 = 6k$$

So we see

$$6k \leq 10$$

and so

$$k \leq 10/6 < 2$$

So there are strictly fewer than 2 connected components, so there must only be one. Thus G is connected.

(In less formal terms, we observed there are 10 vertices total and at least 6 vertices in each connected component, so that there cannot be two or more connected components, as this would require at least 12 distinct vertices.)

Method 2: We show any two vertices in G have a path between them. Let $v, w \in V$ be arbitrary with $v \neq w$.

If $(v, w) \in E$, then v and w have a path of length 1.

Suppose $(v, w) \notin E$. Then v and w do not have edges to each other or themselves. Since $\delta_G(v) \geq 5$ and $\delta_G(w) \geq 5$ and we cannot have multiedges, v is connected to at least 5 of the remaining 8 vertices in $V \setminus \{v, w\}$ (as it cannot connect to itself or w by assumption). Similarly, w is connected to at least 5 of the remaining 8 vertices in $V \setminus \{v, w\}$. By pigeonhole principle, there exists some $x \in V \setminus \{v, w\}$ connected to both v and w. Then $v - x - w$ is a path of length 2 between v and w.

In both cases, there is a path from v to w. Since $v \neq w \in V$ were arbitrary, we conclude G has a path between any two distinct vertices. We conclude G is connected.

Remark: This proof is a bit stronger than method 1, since it not only shows G is connected, but also that the maximum distance between any two vertices is two!

Part b

Through the discussion of method 1 of the previous problem, one might observe that such a graph must have at least 5 vertices in each connected component. So there can still be two connected components of size 5 each. All vertices having degree 4 then enforces that every edge among these groups of 5 must be there. Hence, the only possibility is two disjoint copies of K_5!