# Midterm 1 (Math 61)
## Monday, April 22nd

Circle your discussion section:

| Tuesday | Thursday | |
|---|---|---|
| 2A | 2B | TA: Harris Khan |
| 2C | 2D | TA: Fred Vu |
| 2E | 2F | TA: Matthew Stone |

This exam has 7 pages, including the cover page. Please make sure your exam includes each page. Please write your name on *each* page you submit. You will have 50 minutes to complete this exam. **You may not use a calculator**, or consult your textbook, class notes, or any other materials. If you need scratch paper or more space for your answers, please use the back of the pages.

**If there is any work on the backs of the pages which you would like to have graded, please indicate this clearly on the front of the page for the corresponding problem.**

Show your work for these problems, don't just give an answer. If a question asks you to prove something, please write a complete proof. Unless otherwise stated, you may use any results proved in class or in the textbook, but please make it clear when you are doing so. Unless otherwise stated, you will *not* receive full credit for giving the correct answer with no explanation. You may still earn partial credit even if your final answer is incorrect.

**Remember that you are bound by a conduct code, and that you may not look at anyone's paper or let anyone look at your paper.**

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 35 | |
| 3 | 15 | |
| 4 | 20 | |
| 5 | 20 | |
| Total: | 100 | |

**1.** [10 pts, 2 points each] Mark each of the following statements as either TRUE or FALSE. For this question you do not need to show any work beyond the final answer.

*Be sure to read the questions carefully!*

| | |
|---|---|
| **(a)** The set $\{\{1\}, 3, \{2,3\}, 2, 3\}$ has cardinality 4. | TRUE |
| **(b)** Every relation is either symmetric or antisymmetric. | FALSE |
| **(c)** If $R$ is a relation satisfying $R = R^{-1}$, then $R$ is symmetric. | TRUE |
| **(d)** The set $f = \{(C, \clubsuit), (B, \heartsuit), (E, \heartsuit), (D, \spadesuit), (A, \clubsuit)\}$ is a function from $X = \{A, B, C, D, E\}$ to $Y = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$. | TRUE |
| **(e)** The set $X = \{1, 2, 3, 4, 5, 6\}$ has more subsets of cardinality 4 than subsets of cardinality 3. | FALSE |

2. [35 pts] Compute the following quantities. You may leave your answers in terms of exponents and factorials, but do not leave your final answers in terms of $P(n,r)$ or $C(n,r)$ (so $4^{12}\frac{15!}{3!6!2!}$ would be an acceptable final answer, but $P(10,3)C(18,7)$ would not).

*Show your work. It should be clear how you got your answers.*

(a) [10 pts] The number of permutations of the letters *COMPUTER* that contain the letters *CPU* together in any order (so for instance, *MTUPCREO* would be one such arrangement, but *PMTCOREU* would not).

treating CPU as one item:   [CPU] O M T E R
no letter appear multiple times;
number of permutations = 6!
6! undercounts because it does not consider permutations where CPU appears in other order
for each permutation where CPU appears together, there are 3! orderings of CPU
so the total number is $\boxed{6! \cdot 3!}$

(b) [10 pts] The coefficient of $x^6y^8$ in the expansion of $(3x+2y^2)^{10}$. [Don't forget that $y$ is squared in this expression.]

using the binomial theorem

$$(3x+2y^2)^{10} = \sum_{k=0}^{10} \binom{10}{k}(3x)^k(2y^2)^{10-k}$$

the term corresponding to $x^6y^8$ in the sum is when index $k=6$

the term is $\binom{10}{6}(3x)^6(2y^2)^{10-6}$

$= \binom{10}{6}(3^6)(x^6)(2^4)(y^2)^4$

$= \frac{10!}{6!4!}3^6 2^4 x^6 y^8$

the coefficient is $\boxed{\frac{10!}{6!4!}3^6 2^4}$

(c) [15 pts] A standard 6-sided die (with sides numbered 1-6) is rolled 10 times in a row. How many possible outcomes are there in which *exactly* four 5's were rolled, and no two 5's were ever rolled in a row.

[So for example 4562251565 would be one such outcome, but 2544551533 would not. Also order matters, so 4562251565 would be considered a different outcome from 5254152566.]

possible outcomes that meet the condition will have forms

_ X _ X _ X _ X _ X _ X _   5's will occupy the blank spaces

X represents a non-5 roll   there are 6 X's since 10 rolls − 4 5 rolls = 6 non-5 rolls

number of outcomes $= \binom{7}{4} = \frac{7!}{4!\,3!}$ since there are 7 spaces for 4 5's

each X can have a value 1, 2, 3, 4, or 6

5 possibilities each $\Rightarrow$ total of $5^6$ possibilities for the 6 non-5 rolls

the total number of possibilities is therefore $\boxed{\frac{7!}{4!\,3!}5^6}$

**3.** [15 pts] Let $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$ be the set of positive real numbers. Define a relation $R$ on $\mathbb{R}^+$ by $(x, y) \in R$ if $x/y$ is a rational number. Prove that $R$ is an equivalence relation.

For all $x \in \mathbb{R}^+$, $x/x$ is 1, a rational number, so $(x, x) \in R$

R is reflexive because $(x, x) \in R$ for all $x \in \mathbb{R}^+$

For all $x, y \in \mathbb{R}^+$ such that $(x, y) \in R$, $x/y$ is a rational number and can be written as $p/q$ where $p$ and $q$ are whole numbers; this means $y/x$ can be written as $q/p$ and is therefore also rational, so $(y, x) \in R$.

R is symmetric since whenever $(x, y) \in R$ then $(y, x) \in R$

For all $x, y, z \in \mathbb{R}^+$ such that $(x, y), (y, z) \in R$, $x/y$ and $y/z$ are both rational. This means $x/y$ and $y/z$ can be written as $p/q$ and $r/s$ respectively where $p, q, r$, and $s$ are whole numbers.

$$\frac{x}{y} = p/q, \quad \frac{y}{z} = \frac{r}{s}$$

$$x = \frac{py}{q}, \quad z = \frac{sy}{r}$$

$$x/z = \frac{\frac{py}{q}}{\frac{sy}{r}} = \frac{pr}{qs} \Rightarrow x/z \text{ is rational because it can be written as a quotient of whole numbers}$$

R is transitive since $(x, z) \in R$ whenever $(x, y), (y, z) \in R$.

R is an equivalence relation since it is reflexive, symmetric, and transitive.

**4.** [20 pts] Prove that for any positive integer $n$:

$$1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1$$

[Hint: Use induction.]

base case: $n=1$

$1(1!) = 1(1) = 1$

$(1+1)! - 1 = (2)! - 1 = 2 - 1 = 1$

the hypothesis holds for $n=1$

inductive step:

assuming the hypothesis holds for $n=k$, for $n=k+1$:

$1(1!) + 2(2!) + \cdots + k(k!) + (k+1)(k+1)! = (1(1!) + 2(2!) + \cdots + k(k!)) + (k+1)(k+1)!$

Since by the inductive hypothesis, $1(1!) + 2(2!) + \cdots + k(k!) = (k+1)! - 1$,

$= ((k+1)! - 1) + (k+1)(k+1)!$

$= (k+1)! - 1 + k((k+1)!) + (k+1)!$

$= ((k+1)!)(1 + k + 1) - 1$

$= ((k+1)!)(k+2) - 1$

$= (k+2)! - 1$

$= ((k+1)+1)! - 1 \Rightarrow$ hypothesis holds for $n = k+1$

Since the hypothesis holds for $n=1$ and it holds for $n=k+1$ whenever it holds for $n=k$, the hypothesis holds for all positive integers.

5. [20 pts] Let $X, Y$ and $Z$ be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions, and let $h = g \circ f$ be the composition of $f$ and $g$. (That is, $h$ is the function from $X$ to $Z$ defined by $h(x) = g(f(x))$.)

(a) [10 pts] Prove that if $h$ is one-to-one, then $f$ is one-to-one as well. [Hint: Assume that $f(x) = f(y)$, and try to use that to prove that $x = y$.]

h is one-to-one, so whenever $h(x) = h(y)$, $x = y$

For all $x, y \in X$ such that $h(x) = h(y)$, $f(g(x)) = f(g(y))$

so, $g(x) = g(y)$

whenever outputs of f are equal, the inputs are equal

f is one-to-one since whenever $f(u) = f(v)$, $u = v$

(b) [10 pts] Give a counterexample to show that it is possible for $h$ to be one-to-one while $g$ is not one-to-one. That is, give examples of sets $X, Y$ and $Z$ and functions $f : X \to Y$ and $g : Y \to Z$ such that $h = g \circ f$ is one-to-one, but $g$ is *not* one-to-one.

let $X, Y, Z = \mathbb{R}$