Math 61

Midterm #2

Name SOLUTIONS

Bruin ID ______

Signature ______

TA/Section ______

You have 50 minutes.
There are 5 problems.
Make sure to show all work if you want to get full credit.
No notes, books, calculators, smartphones,... are allowed.

GOOD LUCK!

Question	Points	Your Score
Q1	9	
Q2	12	
Q3	10	
Q4	7	
Q5	12	
TOTAL	50	

Problem 1 (9 points) Solve the following recurrence relation:

$$a_n = -8a_{n-1} - 16a_{n-2}$$

with the initial conditions $a_0 = 2$ and $a_1 = -20$.

Roots of $x^2 + 8x + 16 = 0$

$(x + 4)^2 = 0$ are -4 and -4.

Therefore the solution is of the form

$$a_n = b(-4)^n + d \cdot n \cdot (-4)^n$$

$a_0 = b = 2$

$a_1 = -4b - 4d = -20$

So $\underline{b = 2}$

$b + d = 5$

$d = 5 - 2 = 3$

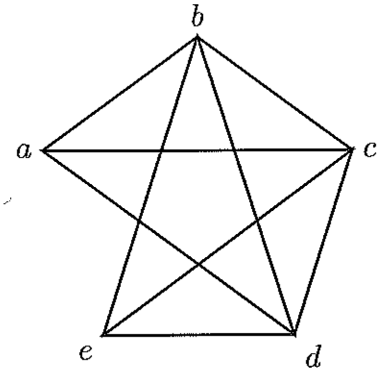
$\underline{d = 3}$

Therefore

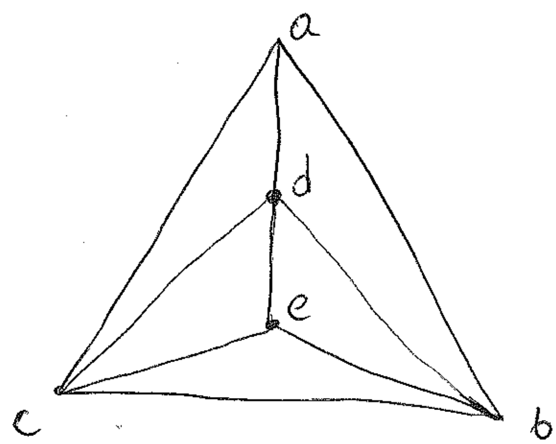
$$\boxed{a_n = 2(-4)^n + 3n(-4)^n}$$

Problem 2 (12 points)

a) (4 points) Determine whether the following graph is planar. If the graph is planar, redraw it so that the edges do not cross. If the graph is not planar, explain why.



The graph is planar:



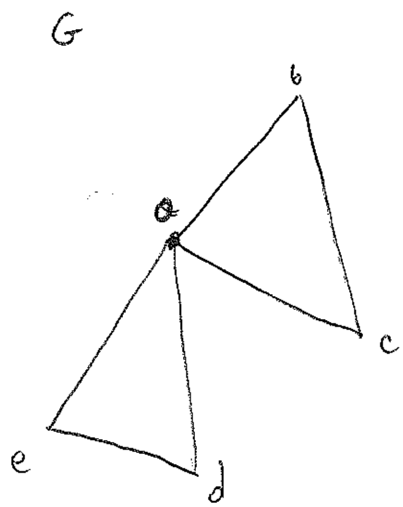
b) (4 points) Is there a simple graph with 6 vertices having degrees 1, 2, 3, 4, 5, 5? If yes, draw such a graph. If no, explain why no such graph exists.

There is no such graph.

Suppose towards the contradiction that there is such a graph. Let v and w be vertices that have degree 5, and let u be the vertex that has degree 1.

Note that v is incident to every vertex (different from v). Similarly for w. In particular, v is incident to u and w is incident to u. But that means that the degree of u is at least 2. A contradiction.

c) (4 points) Give an example of a (connected) graph that has an Euler cycle but it does not have a Hamiltonian cycle.



Clearly, (a, b, c, a, d, e, a) gives an Euler cycle.

Any cycle that contains all vertices, contains a at least twice.

Problem 3 (10 points)
a)(5 points) Suppose there are 100 people at a party and every person shakes a hand with at least one other person (one cannot shake a hand with himself or herself). Show that at the end of the party there are at least two people who have shaken hands with the same number of people.

Each person has shaken hands with 1, 2, 3, ... , or 99 other people. There are 99 possibilitées on the number of people with whom a person has shaken hands ; and there are 100 people.

By the Pigeonhole Principle, there are at least two people who have shaken hands with the same number of people

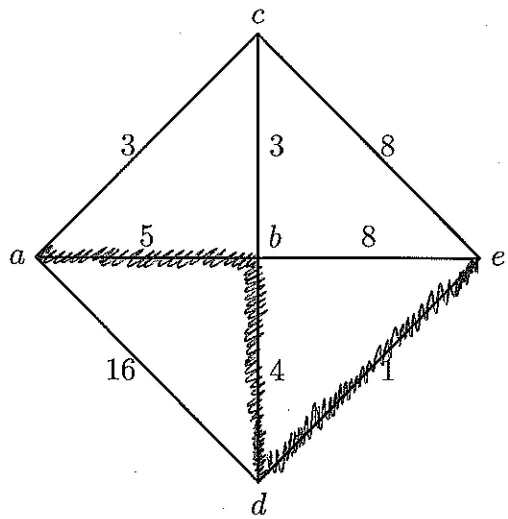
b)(5 points) Show that among any distinct 51 integers from 1,2,3,...,99,100, there are two whose sum is 100.

As clarified during the exam, the 51 numbers are distinct, however, we allow summing two same numbers.

- If among our 51 numbers there is '50', as $50+50=100$, the conclusion is true.

- Otherwise, if there is no '50' among the ~~50~~ 51 numbers chosen, consider "pigeonholes":
$\{1,99\}$, $\{2,98\}$ $\{3,97\}$, ..., $\{49,51\}$, $\{100\}$
There are 51 numbers and 50 "pigeonholes".
Therefore by the Pigeonhole Principle, we have two ~~numbers~~ numbers in the same pigeonhole.
By the choice of pigeonholes, the sum of these two numbers is equal to 100.

Problem 4 (7 points) Using Dijkstra's algorithm, find the length of the shortest path from a to e. Additionally, mark on the graph the shortest path from a to e.



The length of the shortest path from a to e is $5 + 4 + 1 = \underline{10}$

Problem 5 (12 points - 2 points each) Each of the statements below is either true or false. Circle the correct answer. No justification is needed.

(a) K_5 contains a Hamiltonian cycle.
True **False**

(b) There is a simple connected planar graph which has 10 vertices and 30 edges.
True **False**

see exercise 14 in 8.7

(by the way, in that exercise 14, one needs to assume that $v \geq 3$, which they forget to say in the book)

(c) Let $\{F_n\}$ be the Fibonacci sequence. Then $F_6 = 8$.
True **False**

Both answers were accepted; in class we had

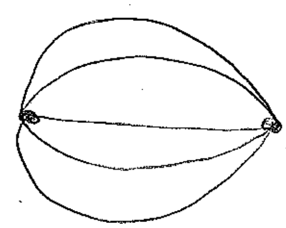
$F_1 = 1$
$F_2 = 1$
$F_n = F_{n-1} + F_{n-2}$

, but it is also possible to consider

$F_0 = 1$
$F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$

(d) There are two isomorphic graphs such that one of them has an Euler cycle and the other does not.
True **False**

(e) There is a graph that has 2 vertices and 5 edges.
True **False**



(f) There is a graph that has 5 edges and 5 vertices having degree 1, 2, 2, 3, 4.
True **False**

$1+2+2+3+4 = 12 \neq 2 \cdot 5$