Math 61

Final

Name SOLUTIONS

Bruin ID

Signature

TA/Section

You have 3 hours.
There are 12 problems.
Make sure to show all work if you want to get full credit.
No notes, books, calculators, smartphones,... are allowed.

GOOD LUCK!

Question	Points	Your Score
Q1	8	
Q2	8	
Q3	12	
Q4	10	
Q5	8	
Q6	8	
Q7	7	
Q8	7	
Q9	8	
Q10	8	
Q11	8	
Q12	8	
TOTAL	100	

Problem 1 (8 points) You have 3 boxes: blue, yellow, and pink. You also have 100 chocolates (chocolates are indistinguishable). In how many ways one can place chocolates in these 3 boxes? Explain your answer.

If you place x_1 many chocolates in the blue box, x_2 chocolates in the yellow box, and x_3 chocolates in the pink box, then $x_1 + x_2 + x_3 = 100$

Also each solution to $x_1 + x_2 + x_3 = 100$ $x_1, x_2, x_3 \geq 0$, corresponds to (since chocolates are indistinguishable) exactly one way of placing chocolates into boxes (x_1 into blue, x_2 into yellow, and x_3 into pink).

Summarizing:

The number of ways in which you can place chocolates into boxes = the number of solutions to $x_1 + x_2 + x_3 = 100$, $x_1, x_2, x_3 \geq 0$, $x_1, x_2, x_3 \in \mathbb{Z}$.

This ~~was~~ is $C(100 + 3 - 1, 3 - 1)$

$= C(102, 2)$.

Problem 2 (8 points) Consider the sequence of real numbers defined by the conditions $x_1 = 1$ and

$$x_{n+1} = \sqrt{1 + 2x_n}$$

for $n \geq 1$. Use the Principle of Mathematical Induction to show that $x_n < 4$ for all $n \geq 1$.

Step 1

$n = 1$

$X_1 = 1$. Clearly, $1 < 4$.

Step $n \to (n+1)$

By the inductive assumption, $x_n < 4$. We have to show that $x_{n+1} < 4$.

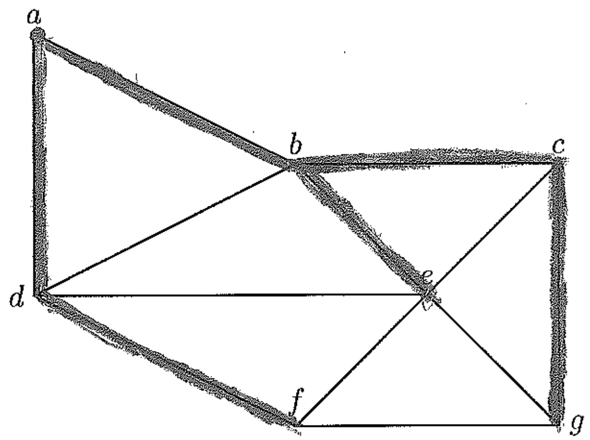
Note that

$$x_{n+1} = \sqrt{1 + 2x_n} < \sqrt{1 + 2 \cdot 4} = \sqrt{9} = 3 < 4.$$

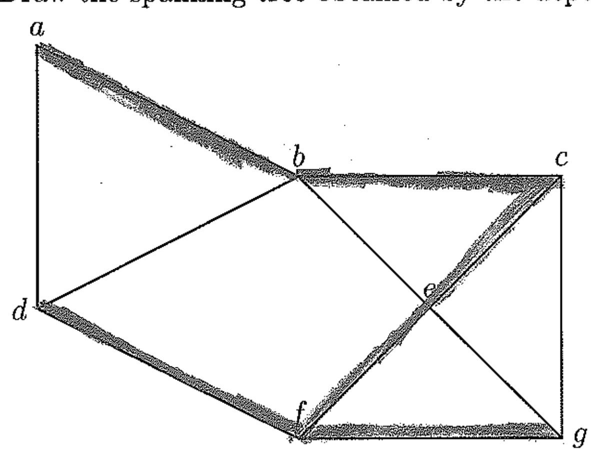
This shows $x_{n+1} < 4$.

Problem 3 (12 points - 4 points each)

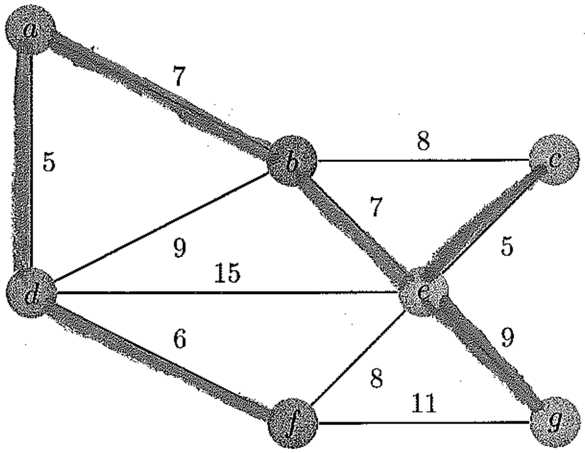
a) Draw the spanning tree obtained by the breadth-first search. Use the alphabetical order on the vertices.



b) Draw the spanning tree obtained by the depth-first search. Use the alphabetical order on the vertices.



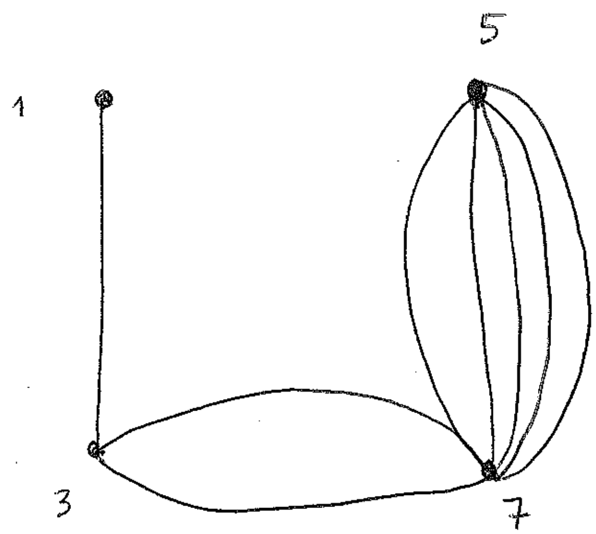
c) Using Prim's algorithm find a minimal spanning tree. Mark on the graph this minimal spanning tree. Start with the vertex a.



Problem 4 (10 points - 5 points each) For each of a) and b) either draw a graph that has the given properties or explain why no such graph exists.

a) Four vertices having degrees 1, 3, 5, and 7.

YES

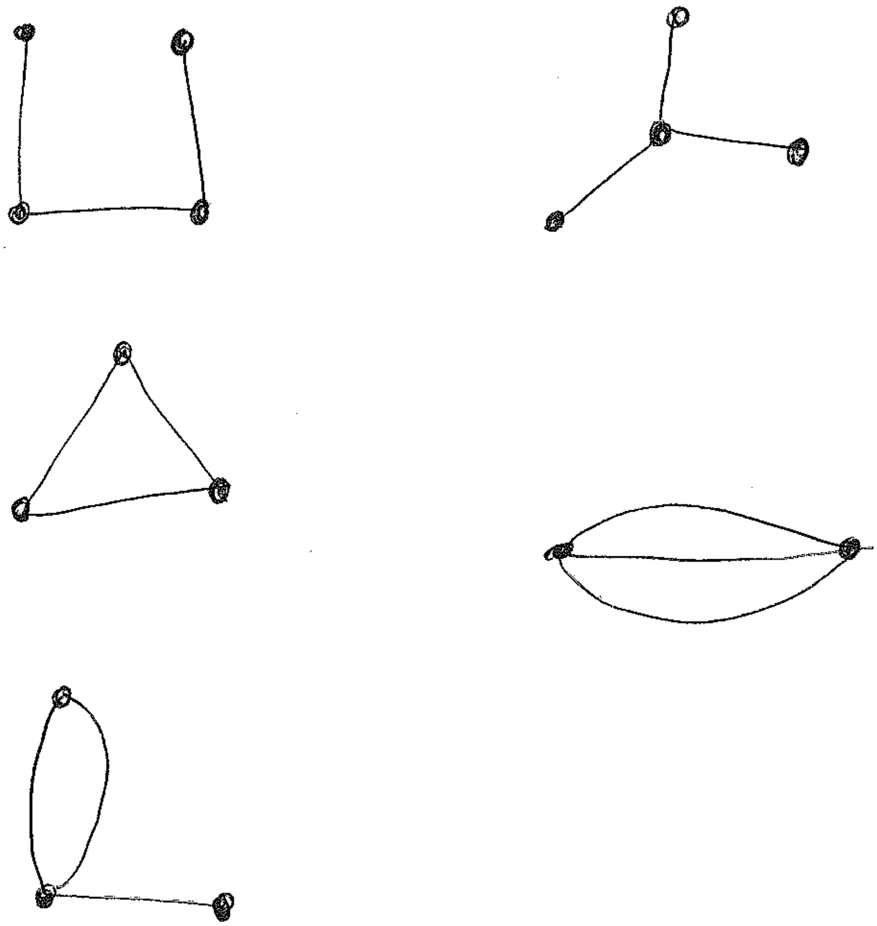


b) Binary tree that has eight vertices and nine edges.

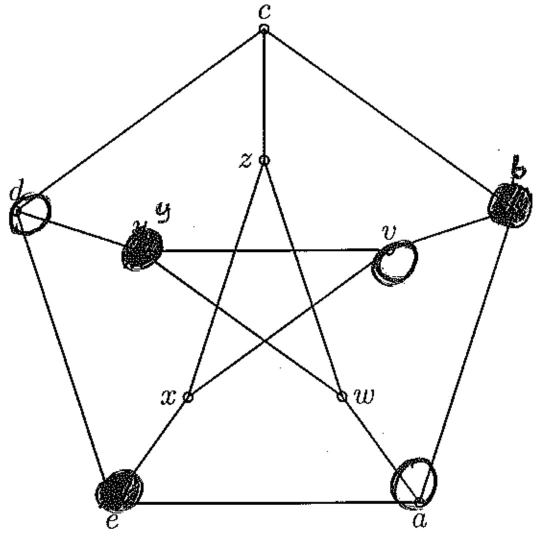
NO

A tree (in particular, a binary tree) with 8 vertices always has 7 edges.

Problem 5 (8 points) Draw all non-isomorphic connected graphs that have 3 edges and have no loops (note: I do not require them to be simple).



Problem 6 (8 points) Determine if the following graph is planar. If the graph is planar, redraw it so that the edges do not cross. If the graph is not planar, explain why.



NO

This graph has a subgraph homeomorphic to $K_{3,3}$

Eliminate edges (z,c), (w,z), and (z,x).

Replace: (d,c) and (c,b) by (d,b)

(y,w) and (w,a) by (y,a)

(e,x) and (x,v) by (e,v)

We obtained $K_{3,3}$

(Say, top vertices : b, e, y

bottom —"— : a, d, v)

Problem 7 (7 points) Let S_n denote the number of n-bit strings that do not contain the pattern 01. Find a recurrence relation and initial conditions for the sequence $\{S_n\}$.

$$S_n = S_n^0 + \underbrace{S_n^1}_{=S_{n-1}} = S_n^0 + S_{n-1} = S_n^{00} + \underbrace{S_n^{01}}_{=0} + S_{n-1}$$

$$= S_n^{000} + \underbrace{S_n^{001}}_{=0} + S_{n-1} = S_n^{0000} + \underbrace{S_n^{0001}}_{=0} + S_{n-1}$$

$$= \dots = S_{n-1} + S_n^{\overbrace{00\dots0}^{n\text{ times}}} = S_{n-1} + 1$$

$$\boxed{S_n = S_{n-1} + 1}$$

(explanation of the notation above:

S_n^0 = # n-element strings that start with 0, (and) don't contain 01

S_n^1 = # — // — // — 1 — // —

S_n^{00} = # — // — // — 00 — // —

S_n^{01} = # — // — // — 01 — // —

etc

The initial condition

$S_1 = 2$

(Remark: you can also start with 0, then you start with $S_0 = 1$)

Problem 8 (7 points) Show that the binary relation R on $\mathbb{R}$ ($\mathbb{R}$ - real numbers) given by xRy if and only if $x^2 = y^2$ is an equivalence relation. What is the equivalence class of 1?

→ R is ~~symmetric~~ reflexive

xRx because $x^2 = x^2$ for any $x \in \mathbb{R}$

→ R is symmetric

for any $x, y \in \mathbb{R}$, if xRy, that is $x^2 = y^2$, then as also $y^2 = x^2$, we have yRx.

→ R is transitive

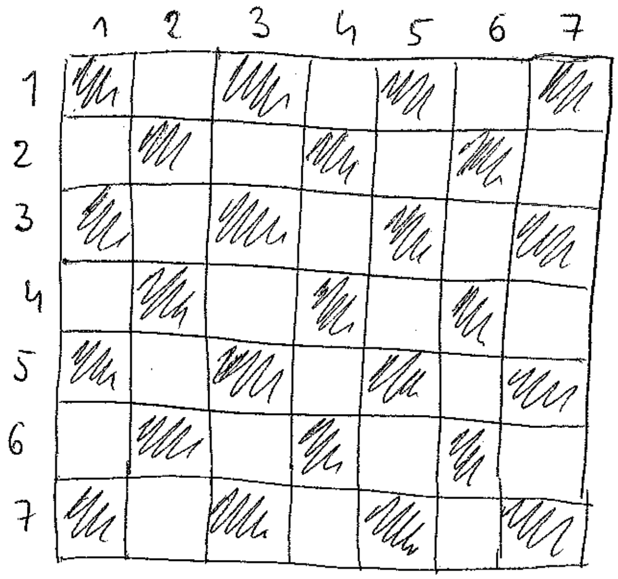
for any $x, y, z \in \mathbb{R}$, if xRy and yRz, that is if $x^2 = y^2$ and $y^2 = z^2$, we get $x^2 = z^2$. This means xRz.

$$[1] = \{x \in \mathbb{R} : xR1\} = \{x \in \mathbb{R} : x^2 = 1^2\}$$
$$= \{-1, 1\}$$

Problem 9 (8 points) You have the 7×7 board $\{1,2,3,4,5,6,7\} \times \{1,2,3,4,5,6,7\}$. Form the following graph: Let the set of vertices be the set of squares of the board (so there are 49 vertices), and let two vertices (a,b) and (c,d) be incident if and only if either ($|a-c|=1$ and $|b-d|=2$) or ($|a-c|=2$ and $|b-d|=1$) (in other words: (a,b) and (c,d) are incident if and only if a knight can move from (a,b) to (c,d)). Show that there is no Hamiltonian cycle in this graph.

In class, we did that problem for a 5×5 board

Color the ~~the~~ board in black and white as below.



Note that if (a,b) and (c,d) are incident then one of these two vertices is black and the other one is white.

Therefore every cycle contains the same number of black and white vertices.

Since the board has 25 black and 24 white vertices, and $25 \neq 24$, there is no cycle that contains all vertices, i.e. there is no Hamiltonian cycle.

Problem 10 (8 points) Peter will work 123 days in 2014, and he will not work on December 31. Show that there will be a day when he works such that he also works either two days later or four days later.

Let $a_1, a_2, a_3, \dots, a_{123}$

be numbers that represent days in which Peter will work.

Consider also numbers

$$a_1+2,\ a_2+2,\ a_3+2, \dots, a_{123}+2$$

and

$$a_1+4,\ a_2+4,\ a_3+4, \dots, a_{123}+4$$

Note that each of the ~~numbers~~ $123 \cdot 3 = 369$ numbers above is ≥ 1 and $\leq 364 + 4 = 368$.

the largest number from $a_1, a_2, \dots, a_{123}$ is ≤ 364 because Peter did not work on December 31, and 2014 has 365 days)

Since $368 < 369$, by the Pigeonhole Principle, there are two numbers that are equal. These two numbers have to come from different lists. We have either $a_i = a_j + 2$ (then a_j is as required)

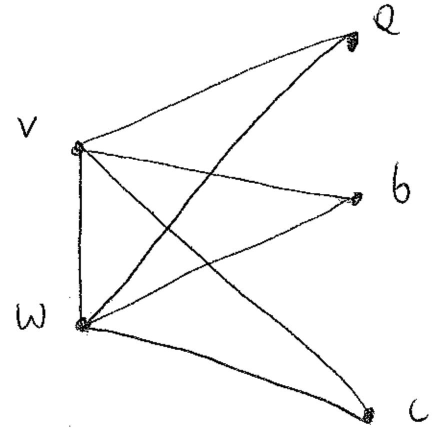
or $a_i = a_j + 4$ (— // — // —)

or $a_i + 2 = a_j + 4$

i.e. $a_i = a_j + 2$ (— // — // —)

Problem 11 (8 points - 4 points each) Let v and w be the two vertices of $K_{3,2}$ that are adjacent to each of the remaining three vertices. Get G from $K_{3,2}$ by adding an edge incident on v and w.
a) Does G have an Euler cycle? Explain carefully your answer.

G it is



YES

Because each vertex has an even degree (either 2 or 4).

b) Does G have a Hamiltonian cycle? Explain carefully your answer.

NO

Suppose towards a contradiction that there is a Hamiltonian cycle C in G.

Since a and b and c have degree 2, edges (v, a), (w, a), (v, b), (w, b) & (v, c), and (w, c) have to be in C.

That means that in C there are at least three edges incident to v. However, v occurs exactly once in C. So in C there have to be exactly 2 vertices incident to v. A contradiction (Same will be true for w.)

Problem 12 (8 points) You have 4 coins: C_1, C_2, C_3, C_4. Exactly one of the coins is heavier or lighter than the rest. Draw a decision tree that gives an algorithm that identifies in at most two weighings the fake coin (but not necessarily determines whether it is heavier or lighter than the other coins) using only a pan balance.