Math 61
Fall 2017
11/20/17
Time Limit: 50 Minutes

Name (Print): ______________________
SID Number: ______________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, circle your section, and put your initials on the top of every page, in case the pages become separated.

Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a result from class, discussion, or homework you must indicate this** and explain why the result may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	2	
2	2	
3	3	
4	3	
5	3	
6	2	
Total:	15	

1. (2 points) The summand not involving x in the expansion of

$$\left(\frac{x}{2} + \frac{k}{x}\right)^6$$

is 20. What is k?

Solution: By the Binomial Theorem, the summand not involving x is

$$\binom{6}{3}\left(\frac{x}{2}\right)^3\left(\frac{k}{x}\right)^3 = \binom{6}{3}\frac{k^3}{2^3} = 20 \cdot \frac{k^3}{2^3}.$$

For this to be equal to 20, k needs to be 2.

2. (2 points) Show that in the decimal expansion of the quotient of two integers, eventually some block of digits repeats. (Examples:

$$\frac{1}{6} = 0.1\underline{6}66\ldots, \qquad \frac{217}{660} = 0.32\underline{87}8787\ldots)$$

Hint: If we divide a by b, the remainder is one of 1,2,...,$b-1$. Consider what happens after b divisions.

Solution: To compute the decimal expansion of $\frac{a}{b}$ we do long division. The important thing to notice is that the remainder obtained in any step determines completely the subsequent digits in the expansion. This implies that if we obtain the same remainder twice the block of digits obtained in between will repeat forever.

Another thing to notice is that if the remainder is 0 the process stops and the digit 0 repeats from there on in the expansion. So we may assume that the remainder is one of $1, 2, \ldots, b-1$. But then the pigeonhole applies to show that in the first b steps of the long division process the same remainder is obtained twice, so the reasoning above allows to conclude.

3. (3 points) Consider the following recurrence relation and initial conditions:

$$U_n = U_{n-1} + 2U_{n-2} + 2n^2 - 10n + 9, \qquad U_1 = 13, \quad U_2 = -12.$$

You may assume that $U_n^{(\mathrm{p})} = -n^2$ is a particular solution to the recurrence relation. Find U_{100}. (No need to simplify your formula.)

Solution: Recall that a solution to the recurrence relation will be given by the sum of homogeneous and particular solution $U_n = U_n^{(\mathrm{h})} + U_n^{(\mathrm{p})}$. So we must find $U_n^{(\mathrm{h})}$.

The associated homogeneous recurrence relation is

$$U_n = U_{n-1} + 2U_{n-2}$$

with characteristic equation

$$r^2 - r - 2 = 0$$

and roots $r_1 = 2$ and $r_2 = -1$. Thus our approach is going to be $U_n^{(\mathrm{h})} = a2^n + b(-1)^n$ for some constants a and b.

The initial conditions impose the equations:

$$13 = U_1 = U_1^{(\mathrm{h})} + U_1^{(\mathrm{p})} = 2a - b - 1$$
$$-12 = U_2 = U_2^{(\mathrm{h})} + U_2^{(\mathrm{p})} = 4a + b - 4$$

from which one easily obtains $a = 1$, $b = -12$. Therefore

$$U_n = 2^n - 12(-1)^n - n^2.$$

In particular,

$$U_{100} = 2^{100} - 12 - 100^2 = 2^{100} - 10012.$$

4. (3 points) In each case below, either give an example of a graph satisfying the stated conditions or explain why there is no such example:

   1. a connected graph with 2 edges which does not admit an Euler cycle;
   2. a simple graph with 4 vertices whose adjacency matrix equals its incidence matrix for some ordering of the vertices and the edges.

Solution:

1. An example is given by the graph $\bullet$ —— $\bullet$ —— $\bullet$. Indeed, it is connected, has 2 edges but since the degree of the first vertex is odd there is no Euler cycle.

2. There is no such example, and students came up with many clever ways to show this. I will give two of these below. Both of them start by assuming that there is such a graph and deduce a contradiction. So let M denote the adjacency matrix which is equal to the incidence matrix. It is a 4×4-matrix. Note that being an incidence matrix of a graph without loops, every column of M has two 0's and two 1's.

   - *First solution:* Consider the first column of M with non-zero entries in the ith and jth spot. Being the first column of the adjacency matrix these tell us that v_1 is adjacent to v_i and v_j. Being the first column of the incidence matrix they also tell us that v_i and v_j are adjacent. Thus v_1, v_i, and v_j form a triangle. There is one vertex and one edge left but this implies that this vertex will have degree at most 1 (as there are no loops). This is a contradiction as the degree of every vertex is 2 (indeed, the sum of the entries in any column is 2).
   - *Second solution:* Consider again the first column of M. One of the 0's is in the first spot since the diagonal of the adjacency matrix consists of 0's. Let's say the other 0 is in the kth spot. Now consider the kth column of M. Since M is symmetric, the first spot is 0. Also, the kth spot is 0 since it is a diagonal entry of M. This implies that the first and the kth column of M are identical. M being the incidence matrix this would imply that the graph has two parallel edges but the graph is simple. Contradiction!

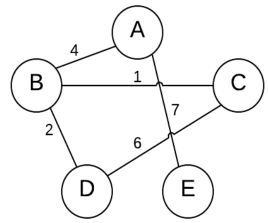
5. (3 points) In this problem, we are concerned with the divisibility relation on the set of integers $\{2, 3, 4, \ldots\}$ given by
$$\{(x, y) \mid x \text{ divides } y\}.$$
Consider the associated digraph, and let G be the underlying (undirected) graph, obtained by forgetting that the edges have a direction. Is G connected? What is the shortest path length for two vertices in the same component? *Hint: the answer to the last question will depend on the two vertices picked.*

Solution: Very concisely, the shortest path length between two integers $m, n \geq 2$ is $|\{m, n, \operatorname{lcm}(m, n)\}| - 1$. In particular, the graph is connected.

To expand on that, given two distinct vertices m, n there are two cases:

- If m divides n or n divides m then the two vertices are adjacent, and the shortest path length is 1.
- Otherwise, a shortest path is given by
$$m \longrightarrow \operatorname{lcm}(m, n) \longrightarrow n\ ,$$
so the shortest path length is 2.

6. (2 points) Write the order in which the shortest path algorithm visits the vertices of the graph when finding the shortest path from C to E. (Here, we say that the algorithm *visits* a vertex when the label of the vertex becomes permanent.)



Solution: In fact, the algorithm always visits the vertices in order of increasing distance from the starting vertex. Therefore the order is as follows (in parentheses the distance from C): C (0), B (1), D (3), A (5), E (12).