Practice Midterm 1

UCLA: Math 61, Winter 2018

Instructor: Jens Eberhardt
Date: 02 February 2017

- This exam has 4 questions, for a total of 34 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.

Name: ______________________________

ID number: ______________________________

Discussion section (please circle):

Day/TA	HAN, KYUTAE	HUNT, CHRISTOPHER	MENEZES, DEAN
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Question	Points	Score
1	8	
2	12	
3	6	
4	8	
Total:	34	

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) Let $X = \{1, 2, 3\}$, then $R = \{(1, 1), (2, 2), (3, 1), (2, 1)\}$ is

   A. transitive, antisymmetric

   B. a function

   C. antisymmetric, reflexive

   D. symmetric, reflexive

   (b) (2 points) Define a partition $\mathcal{P}$ on $\mathbb{Z}_{\geq 0}$ by

   $$\mathcal{P} = \{\{0, 2, 4, 6, \dots\}, \{1, 3, 5, 7, \dots\}\}.$$

   Let $R_{\mathcal{P}}$ be the associated equivalence relation on $\mathbb{Z}_{\geq 0}$. Then

   A. $\mathcal{P} = [23456]_{R_{\mathcal{P}}} \cup [36731]_{R_{\mathcal{P}}}$

   B. $\mathcal{P} = [23456]_{R_{\mathcal{P}}} \cup [36730]_{R_{\mathcal{P}}}$,

   C. $\mathcal{P} = \{[578134]_{R_{\mathcal{P}}}, [578235]_{R_{\mathcal{P}}}\}$

   D. $\mathcal{P} = \{[578134]_{R_{\mathcal{P}}}, [578232]_{R_{\mathcal{P}}}\}$

(c) (2 points) Let $X = \{0, 1, 2, 3, 4\}$. Denote by $\mathcal{P}(X) = \{S | S \text{ is a subset of } X\}$ the power set of X. Then

A. $|\mathcal{P}(X)| = 2^5$
B. $|\mathcal{P}(X)| = 2^4$
C. $|\mathcal{P}(X)| = 4 \cdot 2$
D. $|\mathcal{P}(X)| = 5 \cdot 2$

(d) (2 points) Let $r \neq 1$ be a real number. Define $s_i = r^i$ for $i \geq 0$. Then for $n \geq 1$

$$\sum_{i=0}^{n-1} (5s_i + i)$$

is equal to

A. $5\frac{r^{n+1}-1}{r-1} + \frac{n(n+1)}{2}$
B. $5\frac{n^r-1}{n-1} + \frac{r(r-1)}{2}$
C. $5\frac{n^{r+1}-1}{n-1} + \frac{r(r+1)}{2}$
D. $5\frac{r^n-1}{r-1} + \frac{n(n-1)}{2}$

2. In the following questions, simply write down your answer. There is *no justification needed.* Do not simplify expressions as $2^4, 6!, C(n,r), \ldots$.

   (a) (2 points) Define $f : \mathbb{Z} \to \mathbb{Z}, x \mapsto 4x - 3$. Is f injective? Is f surjective? Is f bijective?

   (b) (2 points) Determine the number of 5-bit strings starting in 001.

   (c) (2 points) Determine the number of 5-bit strings ending in 10.

   (d) (2 points) Determine the number of 5-bit strings starting in 001 or ending in 10.

   (e) (2 points) Assume that you have 50 friends. You want to order your friends by how much you like them. In how many possible ways could you do this?

   (f) (2 points) Assume that you and your 5 best friends want to launch a company. In how many ways could you choose one CEO and two members of the board (in any order).

Solution:

(a) f is injective, not surjective, not bijective.

(b) 2^2

(c) 2^3

(d) $2^3 + 2^2 - 1$

(e) $50!$

(f) $6C(5,2)$

3. (6 points) Prove by induction that

$$\sum_{i=1}^{n}(i+1) = \frac{(n+3)n}{2}$$

for any integer $n \geq 1$.

Solution: Base case $n = 1$: If $n = 1$ we have

$$\sum_{i=1}^{n}(i+1) = (1+1) = 2$$

and

$$\frac{(n+3)n}{2} = \frac{(1+3)1}{2} = 2.$$

Induction step $n \Rightarrow n+1$: Assume that the statement is true for n. We have to show that it is also true for $n+1$.

$$\sum_{i=1}^{n+1}(i+1) = n+2+\sum_{i=1}^{n}(i+1)$$

By the indution hypothesis:

$$\begin{aligned} &= n+2+\frac{(n+3)n}{2} \\ &= \frac{2n+4+(n+3)n}{2} \\ &= \frac{2n+4+n^2+3n}{2} \\ &= \frac{n^2+5n+4}{2} \\ &= \frac{((n+1)+3)(n+1)}{2} \end{aligned}$$

Hence the statement is true by induction.

4. Answer the following questions, *justifying* your answers. (If an answer is Yes, explain why. If an answer is No, give a counterexample.)

   (a) (2 points) Define a relation R on $\mathbb{Z}$ by
   $$xRy \text{ if } y - 3 = x.$$
   Is R a function?

   (b) (2 points) Let X, Y be finite sets. Then
   $$|X \cap Y| = |X| + |Y| + |X \cup Y|.$$

   (c) (2 points) Let $X = \{1, 2, 3 \dots\}$ and $Y = \{2, 4, 6, \dots\}$. Show that there is a bijection between X and Y.

   (d) (2 points) Let X be a set and R a relation on X, which is transitive and symmetric. Is R reflexive?

Solution:

(a) Yes, since for every $x \in \mathbb{Z}$, there is a unique $(x, y) \in R$, namely $(x, x + 3)$.

(b) No. Consider $X = \{1, 2\}$ and $Y = \{2, 3\}$. Then $|X \cap Y| = 1$ and $|X|+|Y|+|X \cup Y| = 2+2+3 = 7$.

(c) Yes. Define $f : X \to Y$ by $f(x) = 2x$. Then f is injective and surjective, hence bijective.

(d) No. Let $X = \{1\}$ and $R = \emptyset$. Then R is symmetric and transitive but $(1, 1) \notin R$. Hence R is not reflexive.

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