061 - Final - Practice Problems

1 June 2011

1. Prove that $n! > n^2$ for all integers $n \geq 4$.

Solution Proof by induction. Base case: when $n = 4$, $n! = 24$, while $n^2 = 16$, so this checks out. Now, suppose that $n > 4$ and that the statement is true for all k where $4 \leq k < n$. Then,

$$n! = n((n-1)!) > n((n-1)^2) = n^3 - 2n^2 + n = n^2(n-2) + n > n^2,$$

as desired.

2. Let X be a finite set with n elements. Determine, with proof, how many binary equivalence relations there are on X.

Solution A binary relation on X is just a subset of $X \times X$. The subsets of $X \times X$ are the elements of the power set $P(X \times X)$. The set $X \times X$ has n^2 elements, so the set $P(X \times X)$ has $2^{(n^2)}$ elements. Therefore, there are $2^{(n^2)}$ binary relations on X.

3. How many rearrangements of MATHEMATICS are there where the Ms are not next to each other?

Solution In general, there are a total of

$$\frac{11!}{2!2!2!}$$

rearrangements of MATHEMATICS. Let $\Phi =$MM. Then, there are

$$\frac{10!}{2!2!}$$

rearrangements of ΦATHEATICS. These correspond to the rearrangements of MATHEMATICS in which the Ms *are* next to each other. So, there are

$$\frac{11!}{2!2!2!} - \frac{10!}{2!2!}$$

rearrangements of MATHEMATICS where the Ms are not next to each other.

4. Let's play Canasta! The deck consists of 2 standard packs of 52 cards, 13 in each of 4 suits. So, there are 2 of every card, but we can't tell the two copies apart. For example, there are 2 Aces of Hearts. How many different 5-card hands are there that contain only Hearts?

Solution First, suppose that the hand contains no duplicates; e.g., there are not 2 Aces of Hearts in the hand. Then, there are $\binom{13}{5}$ such hands. Now, suppose that a single card is duplicated. There are 13 choices for the duplicated card, and $\binom{12}{4}$ choices for the other cards. If 2 cards are duplicated, there are $\binom{13}{2}$ choices for those cards and $\binom{11}{1}$ choices for the other card. Therefore, there are

$$\binom{13}{5} + \binom{13}{1}\binom{12}{4} + \binom{13}{2}\binom{11}{1}$$

different flushes of Hearts.

5. Let $X = \{1, 2, 3, 4, 5\}$. How many strings of length 1000 on X are there such that there are no substrings from $\{1, 2\}$ of length more than 1.

Solution Let a_n be the number of string of length n on X such that there are no substring from $\{1, 2\}$ of length more than 1. Then, $a_0 = 1$ and $a_1 = 5$. We find a recursive formula for the a_n. Given any string t of length $n-1$ on X of the same type, the strings $3t$, $4t$, and $5t$ are all of the appropriate type. Similarly, given any string t of length $n-1$ on X of this type, the strings $13t$, $14t$, $15t$, $23t$, $24t$, and $25t$ are of the correct type. Thus, we see that

$$a_n = 3a_{n-1} + 6a_{n-2}.$$

To solve this, we consider the equation $t^2 - 3t - 6$. Using the quadratic formula, this has solutions $r_1 = \frac{3+\sqrt{33}}{2}$ and $r_2 = \frac{3-\sqrt{33}}{2}$. Solving the system of equations

$$\begin{aligned} a + b &= 1 \\ ar_1 + br_2 &= 5, \end{aligned}$$

we find that $a = \frac{7}{\sqrt{33}}$ and $b = 1 - \frac{7}{\sqrt{33}}$. Therefore, there are

$$\frac{7}{\sqrt{33}}\left(\frac{3+\sqrt{33}}{2}\right)^{1000} + \left(1 - \frac{7}{\sqrt{33}}\right)\left(\frac{3-\sqrt{33}}{2}\right)^{1000}$$

such strings.

6. Prove that in any set of 51 positive integers less than 100, there are two whose sum is 100.

Solution Let $a_1, \ldots, a_{51}$ be 51 positive integers less than 100. Let $b_n = 100 - a_n$, for $1 \le n \le 51$. First, note that $b_n = a_n$ if and only if $a_n = 50$. If some a_n is equal to 50, then discarding a_n and b_n, the rest of the numbers form 100 integers between 1 and 99. Thus, two of them are equal by the pigeonhole principle. So, $a_k = b_j = 100 - a_j$ for some $k \neq j$. So, we're done. If no a_n is equal to 50 then the same argument works.

7. Show that if G is a simple graph, then either G or $\overline{G}$ is connected.

Solution Assume that G is a simple disconnected graph. Let $v_1, \ldots, v_k$, $k \ge 2$, be a vertex from each connected component of G. This means that every vertex of G can be connected to exactly one of the v_i, and no v_i can be connected to any other. Let x and y be two vertices in the vertex set of G. We show that they are connected by a path in $\overline{G}$. First, if x and y are in different components in G, then there is actually an edge between them in $\overline{G}$, so they are certainly connected by a path in this case. Now, assume that x and y are in the same component of G, say the v_1 component. Then, there is an edge e_1 from x to v_2 in $\overline{G}$ and an edge e_2 from y to v_2 in $\overline{G}$. Thus, the path (x, e_1, v_2, e_2, y) in $\overline{G}$. Therefore, in this case too x and y are connected. Therefore, $\overline{G}$ is connected.

8. Show that if G is a simple graph with at least two vertices, then there are two vertices in G with the same degree.

Solution Suppose that G has n vertices. Since G is simple, the degree of each vertex is between 0 and $n-1$. If the graph is connected, then the degree of each vertex is between 1 and $n-1$. By the pigeonhole principle, two vertices have the same degree. If the graph is not connected, there is no vertex of degree $n-1$. Thus, the degree of each vertex is between 0 and $n-2$. Again, by the pigeonhole principle, two vertices have the same degree.

9. Prove that every tree with at least two vertices is a bipartite graph.

Solution Choose a root for the tree T. Then, let X consist of the vertices of even level, and let Y be the vertices of odd level. Then, T is bipartite on X and Y.

10. Prove that the number of nonisomorphic binary trees with n vertices is the nth Catalan number.

Solution Denote by C_n this number. Then, C_0 is 1. We can construct all isomorphism classes of binary trees with n vertices by choosing the number of vertices k of the left branch of the root together with a binary tree on k vertices together with a binary tree on $n-k-1$ vertices. Therefore,

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}.$$

But, this is the same recurrence relation satisfied by the Catalan numbers with the same initial condition.