1. (20 points). Let F_n be the nth Fibonacci number. Thus, $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$ for $n > 2$. Show using induction that the statement $P(n)$:

$$(F_{n+1})^2 - F_n F_{n+2} = (-1)^n$$

is true for all $n \geq 0$. Show all your steps and use complete sentences.

First, we check the base case $n=1$. Then, we want to show that

$$F_2^2 - F_1 F_3 = -1.$$

But, $F_3 = 2$, so

$$F_2^2 - F_1 F_3 = 1^2 - 1 \cdot 2 = -1.$$

To show the induction step, assume the equation

$$(F_{n+1})^2 - F_n F_{n+2} = (-1)^n.$$

We want to show $(F_{n+2})^2 - F_{n+1} F_{n+3} = (-1)^{n+1}$.

But,

$$(F_{n+2})^2 - F_{n+1} F_{n+3} = F_{n+2}^2 - F_{n+1}(F_{n+1} + F_{n+2})$$

$$= F_{n+2}^2 - F_{n+1}^2 - F_{n+1} F_{n+2}$$

$$= F_{n+2}(F_{n+2} - F_{n+1}) - F_{n+1}^2 \quad (\text{using } F_n + F_{n+1} = F_{n+2})$$

$$= F_{n+2} F_n - F_{n+1}^2$$

$$= -(-1)^n = (-1)^{n+1},$$ as desired.

2. (20 points). Let $X = \{1,2,3\}$ and $Y = \{a,b,c\}$. Show that if a function $f : X \to Y$ is one-to-one, then it is onto.

Definition. $f : X \to Y$ is one-to-one if

$f(x) = f(y) \Rightarrow x = y$. Equivalently,

f is one-to-one ~~is equal~~

if for every $y \in Y$ there exists at most one solution to the equation

$f(x) = y$, $x \in X$.

From the definition, if $|X|$ is finite,

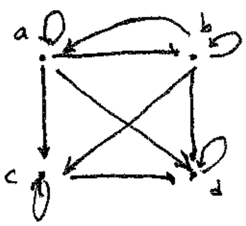
then $|\text{range}(f)| = |X| = 3$. So, range$(f)$ is a subset of Y of order 3.

Thus, it contains a, b, c, and f is onto.

3. Let $X = \{a, b, c, d\}$, and let R be the relation

$$\{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d),(b,a),(b,c),(b,d),(c,d)\}$$

3.a. (10 points). Carefully draw the digraph of R. Label the vertices, but not the edges.



3.b. (10 points). Which of the following properties does R have? (Just circle them.)

- reflexive,
- symmetric,
- transitive.

4. Consider the string COMBINATORICS.

4.a. (10 points). How many reorderings of the string are there?

$$\frac{13!}{2!\,2!\,2!}$$

(Theorem 6.3.2 in book)

4.b. (10 points). How many reorderings of the string are there where no two C's are adjacent?

Consider

_ O _ M _ B _ I _ N _ A _ T _ O _ R _ I _ S _

A re-ordering can be constructed as:

#1 choose two of the 12 blank spots to insert the Cs.

$$\binom{12}{2}$$

#2 choose a re-ordering of OMBINATORIS.

$$\frac{11!}{2!\,2!}$$

TOTAL: $\binom{12}{2}\frac{11!}{2!\,2!}$.

5. (20 points). Consider a deck of 52 cards consisting of 4 suits of 13 cards each. Suppose that each suit has a unique card labeled A, the ace if you like. How many 5 card hands contain exactly two suits and exactly one ace?

Step 1: Pick a suit (Suit 1) to take an Ace out of

$\binom{4}{1}$.

Step 2: Pick another suit (Suit 2): $\binom{3}{1}$.

Step 3: pick a hand of the two suits not containing the ace of the second suit.

if there are no other cards from suit 1, we get

$\binom{12}{4}$

choices (12 because we remove the ace from suit 1).

1 more card from suit 1, 3 from suit 2 $\Big\}$ $\binom{12}{1}\binom{12}{3}$

2 suit 1, 2 suit 2 $\Big\}$ $\binom{12}{2}\binom{12}{2}$

3 suit 1, 1 suit 2 $\Big\}$ $\binom{12}{3}\binom{12}{1}$

Total :

$$\binom{4}{1}\binom{3}{1}\left[\binom{12}{4}+\binom{12}{1}\binom{12}{3}+\binom{12}{2}\binom{12}{2}+\binom{12}{3}\binom{12}{1}\right].$$