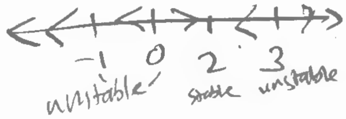
1. (10 points)

a. Consider the equation $y' = y(y+1)^2(y-2)(y-3)$. Find the equilibrium solutions to this equation and determine whether they are asymptotically stable, semi-stable, or unstable.



Equilibrium Solutions

$y=-1$ semistable

$y=0$ unstable

$y=2$ Stable

$y=3$ unstable

b. Suppose that you earn interest from an investment at a rate of .05 (the interest rate is 5% and unit time is one year). You take \$100 per year from the account and you draw the funds continuously from the account. If $y(t)$ denotes the amount of money you have in the account at time t, then $y' = (.05)y - 100$. How much money must you initially have in the account so that the amount of money does not change in time? What (roughly) happens if you start with less than this amount?

No change $\Rightarrow y' = 0$

$0.05y - 100 = 0$

$y = \frac{100}{0.05}$

$y = 2{,}000$

You would need \$2,000 initially in the account so that the amount of money doesn't change. If you have less than that, the money in your account will decrease to zero.

2. (10 points) Solve the initial value problems given below.

a)

5

$$\begin{cases} \frac{dy}{dt} = 2ty(1-y) \\ y(0) = 1. \end{cases}$$

Partial fractions

$\frac{a}{y} + \frac{b}{1-y} = \frac{1}{y(1-y)}$

$a(1-y) + by = 1$

$y=1 \Rightarrow b=1$

$y=0 \Rightarrow a=1$

$\int \frac{1}{y(1-y)} dy = \int 2t\, dt$

$\int \frac{1}{y} + \frac{1}{1-y} dy = \int 2t\, dt$

$\ln|y| - \ln|1-y| = t^2 + C$

$\frac{y}{1-y} = Ae^{t^2}$

$y = Ae^{t^2} - yAe^{t^2}$

$y(1+Ae^{t^2}) = Ae^{t^2}$

$y = \frac{Ae^{t^2}}{1+Ae^{t^2}} = \frac{1}{1+Ae^{-t^2}}$

$y(0) = 1 \Rightarrow A=0$

so $y(t) = 1$ ✓

b)

4

$$\begin{cases} \frac{dy}{dt} + 2y = \cos(t) \\ y(0) = 0. \end{cases}$$

$\frac{dy}{dt} = \cos t - 2y$

$y = A\cos t + B\sin t$

$-A\sin t + B\cos t + 2A\cos t + 2B\sin t = \cos t$

$B + 2A = 1 \qquad -A + 2B = 0$

$B + 4B = 1 \qquad A = 2B$

$B = 1/5 \Rightarrow A = 2/5$

$y(t) = \frac{2}{5}\cos t + \frac{1}{5}\sin t + C$

$y(0) = 0 \Rightarrow 0 = \frac{2}{5} + C$

$C = -2/5$

$y(t) = \frac{2}{5}\cos t + \frac{1}{5}\sin t - \frac{2}{5}$

3. (10 points)

a) The form $\omega = (y\cos(xy) + 2xy + 1)\,dx + (x\cos(xy) + x^2 + y)\,dy = 0$ is exact. Check that it is exact using the appropriate test and then find a function F such that ~~$\omega = dF$.~~

If exact the $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$F = \sin(xy) + x^2y + x + \frac{1}{2}y^2 + C$

$\frac{d}{dy}\left(y\cos(xy) + 2xy + 1\right) = \frac{d}{dx}\left(x\cos(xy) + x^2 + y\right)$

$\cos(xy) - xy\sin(xy) + 2x = \cos(xy) - xy\sin(xy) + 2x$

Find F:

$\int y\cos(xy) + 2xy + 1\,dx = \sin(xy) + x^2y + x + h(y)$

$\frac{d}{dy}(\quad) = x\cos xy + x^2 + y$

$x\cos x + \cancel{x} + h'(y) = x\cos(xy) + x^2 + y$

$h'(y) = y$

$h(y) = \frac{1}{2}y^2 + C$

b) Suppose that $\omega = (x^2 - y^2)\,dx - xy\,dy$. Use the fact that ω is a homogeneous form to find the solution to the equation $\omega = 0$.

$(x^2 - y^2)\,dx - xy\,dy = 0$

$\frac{\partial P}{\partial y} = -2y \quad \frac{\partial Q}{\partial x} = -y$

$h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$h = \frac{1}{-xy}(-y)$

$h = \frac{1}{x}$

$(x - \frac{y^2}{x})\,dx - y\,dy = 0$

+2

$(x^2 - y^2)\,dx = xy\,dy$

$\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$

$\frac{1}{2}xy^2 = \frac{1}{3}x^3 - xy^2$

$\frac{3}{2}xy^2 = \frac{1}{3}x^3$

$y^2 = \frac{2}{9}x^2$

$y = \frac{\sqrt{2}}{3}x$

~~Let~~ $y = vx$

4. (10 points) 10

a) Find the general solution of the equation $x'' + 4x' + 3x = 0$.

Let $x = e^{\lambda t}$

$\lambda^2 + 4\lambda + 3 = 0$

$(\lambda+1)(\lambda+3) = 0$

$\lambda = -1, -3$

$x(t) = c_1 e^{-t} + c_2 e^{-3t}$

b) Find the general solution of the equation $x'' - 2x' + \cancel{x} = 0$.

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1, 1$

$x(t) = c_1 e^{t} + c_2 t e^{t}$

c) Find the general solution of the equation $x'' + x' + x = 0$.

$\lambda^2 + \lambda + 1 = 0$

$\lambda = \frac{-1 \pm \sqrt{1-4}}{2}$

$\lambda = \frac{-1 \pm i\sqrt{3}}{2}$

$\lambda = -\frac{1}{2} \pm \frac{i}{2}\sqrt{3}$

$e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$

$x(t) = e^{-\frac{1}{2}t}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$

5. (10 points) Suppose that D is a second order, linear, constant coefficient differential operator. The associated homogeneous differential equation is given by $Dy = 0$ and has linearly independent solutions y_1 and y_2. You are to use the method of undetermined coefficients to find a particular solution to the differential equation $Dy = f$. Given the information provided in each of the parts below, write down your guess of the form of the particular solution. Do not attempt to figure out the coefficients, rather leave them as variables.

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2 a) $y = y_h + y_p$ Guess: $y_p = Ae^{3t} + Bt^2 + Ct + D$

$$\begin{cases} y_1 = e^{2t} \\ y_2 = e^{-t} \\ f(t) = e^{3t} + t^2 + 1 \end{cases}$$

$y_p = Ae^{3t} + Bt^2 + Ct + D$ ✓

1 b)

$$\begin{cases} y_1 = e^{2t} \\ y_2 = e^{-t} \\ f(t) = \sin(t) + e^{-t} \end{cases}$$

$y_p = A\sin(t) + B\cos(t) + Ce^{-t}$ ✓ ✗

(-1)

0 c)

$$\begin{cases} y_1 = e^{t} \\ y_2 = te^{t} \\ f(t) = (t^2+2)e^{3t} + (t+1)e^{t} \end{cases}$$

$y_p = (At^2 + \cancel{B})Ce^{3t} + (Dt + E)Fe^{t}$ ✗

(-2)

2 d)

$$\begin{cases} y_1 = \sin(t) \\ y_2 = \cos(t) \\ f(t) = e^{t} + \sin(2t) \end{cases}$$

$y_p = Ae^{t} + B\sin(2t) + C\cos(2t)$ ✓