Name: ______________________________ UCLA ID: __________

Please circle your section:

Yin/Tuesday (2A)	Marshak/Tuesday (2C)	Baron/Tuesday (2E)
Yin/Thursday (2B)	Marshak/Thursday (2D)	Baron/Thursday (2F)

Instructions:

1. *Solve all problems completely.* Give complete reasoning unless it is stated that you don't need to.
2. *Show all your work.* Partial credit will be given where it is possible to see that you have a partial answer.
3. No calculators are allowed, or necessary.
4. No notes or references (including but not limited to the textbook) are allowed.
5. You must answer the questions entirely yourself.
6. If you need more space to work, please continue each problem on the back of the same sheet.
7. *Look at all the problems before starting!* You may find that some seem easier, or parts of them; do those first.

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total:	60	

1. (12 points) Mark each statement as "true" or "false"; one point each, and work will not be taken into account. (You may use the blank space below for your notes, though.)

T	**F**	The differential equation $x(y')^2 + y = x^2y^2$ is a second-order differential equation.
T	**F**	Every linear differential equation is separable.
T	**F**	The differential equation $x(3+y)y' + y = 0$ is linear.
T	F	The differential equation $x(3+y)\,dy + y\,dx = 0$ has an integrating factor $u(x,y) = 1/(xy)$.
T	F	If $y = f(x)$ is any solution to the differential equation $2y' + 3y = 1$, then its interval of existence is $\mathbb{R} = (-\infty, \infty)$.
T	F	It is impossible for integral curves of a differential equation $P(x,y)\,dx + Q(x,y)\,dy = 0$ to cross.
T	F	The differential equation $y' = xy$ is homogeneous.
T	**F**	The equation $y' + 2xy^2 = 0$ has the general solution $y_h = 1/(x^2 + C)$ and the equation $y' + 2xy^2 = 2x$ has a particular solution $y_p = 1$; therefore $y' + 2xy^2 = 2x$ has the general solution $y = y_h + y_p = 1/(x^2 + C) + 1$.
T	F	The differential-form equation $(x^2 + xy^2)\,dx + (x^2y - y^2)\,dy = 0$ is exact.
T	**F**	The differential-form equation $x^3y\,dx + xy^2\,dy = 0$ is homogeneous.
T	F	The equation $\sin(x/y)\,dx + \cos(x/y)\,dy = 0$ can be made separable using a change of variables.
T	**F**	Every equilibrium point of the equation $y' = (x-2)^2(x-3)^2$ is stable.

2. Consider the equation

$$y^2\,dx - (x^2 + 2xy)\,dy = 0.$$

(a) (4 points) Show that it is not separable and, using a change of variables, transform it into a separable equation. (Don't solve it.)

Solution: To separate this equation, we would need the $Q(x, y)\,dy$ part to be a product of terms involving only x and terms involving only y. However, it is only a product $x(x + 2y)$, so cannot be separated that way.

The left-hand side is homogeneous, so the substitution $y = vx$ will make it separable:

$$\begin{aligned}(vx)^2\,dx - (x^2 + 2x \cdot vx)(x\,dv + v\,dx) &= v^2x^2dx - (x^2 + 2vx^2)x\,dv - (x^2 + 2vx^2)v\,dx \\ &= (-v^2x^2 - vx^2)\,dx - (x^3 + 2vx^3)\,dv = 0\end{aligned}$$

and canceling the $-x^2$ factor gives the final result:

$$(v + v^2)\,dx + (x + 2vx)\,dv = 0.$$

(Note that this is separable: the second term is $x(1 + 2v)$.)

(b) (4 points) Show that it is not exact and, using an integrating factor $\mu = \mu(x)$, transform it into an exact equation. (Don't solve it.)

Solution: To see that it is not exact we just apply the criterion:

$$D_y(y^2) = 2y \qquad\qquad D_x\big(-(x^2 + 2xy)\big) = -2x - 2y,$$

which are different. The differential equation for an integrating factor $\mu(x)$ is

$$\mu' = \frac{1}{Q}\big(D_y(P) - D_x(Q)\big)\mu = \frac{2x + 4y}{-(x^2 + 2xy)} = -\frac{2}{x}\mu.$$

Solving, we get $\mu = x^{-2}$. Finally, multiplying the original equation by this gives:

$$\frac{y^2}{x^2}\,dx - \left(1 + 2\frac{y}{x}\right)\,dy = 0,$$

which is exact (both partials are $2y/x^2$).

(c) (4 points) Solve the equation using the result of part (b).

Solution: Since it's exact we integrate the terms individually:

$$F(x,y) = \int \frac{y^2}{x^2}\,dx = -\frac{y^2}{x} + f(y)$$
$$D_yF(x,y) = -\frac{2y}{x} + f'(y) \stackrel{?}{=} -1 - 2\frac{y}{x}$$
$$f'(y) = -1 \implies f(y) = -y + C.$$

Therefore the general solution is $F(x,y) = -y^2/x - y = C$, or

$$y^2 + y = Cx.$$

3. Consider the differential equation

$$(3t^2 - 6)y' + 12ty = 2t.$$

(a) (3 points) Find the general solution $y_h(t)$ to the associated homogeneous equation.

Solution: The homogeneous equation is obtained by forgetting the terms not involving y:

$$(3t^2 - 6)y_h' + 12ty_h = 0.$$

It is separable, so can be solved by integration:

$$\frac{dy_h}{y_h} = -\frac{12t}{3t^2-6}\,dt \implies \ln|y_h| = \int -\frac{4t}{t^2-2}\,dt = -2\ln\left|t^2-2\right| + C$$
$$\implies y_h = \frac{C}{(t^2-2)^2} \quad (C \in \mathbb{R}).$$

(b) (5 points) Find the general solution $y(t)$ by any of the following techniques: variation of parameters; finding a particular solution y_p; integrating factor.

Solution: Variation of parameters: $y = v(t)/(t^2-2)^2$, where v satisfies the equation

$$(3t-6)\frac{1}{(t^2-2)^2}v' = 2t \iff v' = \frac{2}{3}t(t^2-2).$$

This can be integrated directly, giving:

$$v = \int \frac{2}{3}t(t^2-2)\,dt = \frac{1}{6}(t^2-2)^2 + C.$$

The general solution is therefore

$$y = \frac{1}{6} + \frac{C}{(t^2-2)^2}.$$

Particular solution: You could guess that $y_p(t) = 1/6$ solves the equation, as you can check:

$$(3t^2-6)y_p' + 12ty_p = 0 + 12t\frac{1}{6} = 2t.$$

Then the general solution is $y = y_h + y_p = 1/6 + C/(t^2-2)^2$, as before. (This is not the intended method, since it requires quite a bit of good luck.)

Integrating factor: In normal form, the equation is

$$y' = \frac{2t}{3t^2-6} - \frac{4t}{t^2-2}y,$$

so the formula for the integrating factor is

$$u = \exp\left(-\int -\frac{4t}{t^2-2}\,dt\right) = \exp\bigl(2\ln\left|t^2-2\right|\bigr) = (t^2-2)^2.$$

Then the solution to the equation is

$$uy = \int (t^2-2)^2 \cdot \frac{2t}{3t^2-6}\,dt = \int \frac{2}{3}t(t^2-2)\,dt = \frac{1}{6}(t^2-2)^2 + C,$$

and finally $y = 1/6 + C/(t^2-2)^2$ again.

(c) (4 points) Find the solution with initial value $y(0) = 1$, and determine its interval of existence.

Solution: No matter what, if we want $y(0) = 1$, we need

$$1 = \frac{1}{6} + \frac{C}{(-2)^2} \implies C = \frac{10}{3}.$$

The specific solution is therefore

$$y = \frac{1}{6} + \frac{10}{3(t^2-2)^2},$$

which is discontinuous at $t = \pm\sqrt{2}$. Since the interval of existence must contain $t = 0$, it is therefore $(-\sqrt{2}, \sqrt{2})$.

4. A sponge has a volume of $30\,\text{cm}^3$ and is initially soaked with pure water. A flow of 33 % (1/3) salt water is directed onto the sponge at a rate of $5\,\text{cm}^3/\text{s}$. As a result of the excess internal pressure, the sponge changes in two ways: first, it begins to expand, but as it is a sponge, it expands only 1/4 as much as the additional volume it absorbs; second, it begins to leak, dripping out $2\,\text{cm}^3/\text{s}$ of fluid.

(a) (5 points) What is the differential equation governing the mass of salt in the sponge?

Solution: Let $x(t)$ be the mass of salt function. Then we have the rates in and out:

$$x'_{\text{in}} = \frac{1}{3} \cdot 5 \qquad\qquad x'_{\text{out}} = \frac{x}{\text{volume}} \cdot 2 = \frac{2x}{30 + \frac{5}{4}t}.$$

(the rate of increase of the volume is 5/4 because the incoming volume is 5 and the expansion factor is 1/4). This gives the equation

$$x' = \frac{5}{3} - \frac{2}{30 + \frac{5}{4}t}x.$$

(b) (4 points) Find the mass of salt function for this sponge.

Solution: The equation is linear, and already in normal form, so we use an integrating factor:

$$u = \exp\left(-\int -\frac{2}{30 + \frac{5}{4}t}\,dt\right) = \exp\left(2 \cdot \frac{4}{5} \ln\left|30 + \tfrac{5}{4}t\right|\right) = (30 + \tfrac{5}{4}t)^{8/5}.$$

The general solution is therefore:

$$\begin{aligned} ux &= \int (30 + \tfrac{5}{4}t)^{8/5} \frac{5}{3}\,dt = \frac{5}{3}\frac{4}{5}\frac{5}{13}(30 + \tfrac{5}{4}t)^{13/5} = \frac{20}{39}(30 + \tfrac{5}{4}t)^{13/5} + C \\ x &= \frac{20}{39}(30 + \tfrac{5}{4}t) + \frac{C}{(30 + \frac{5}{4}t)^{8/5}}. \end{aligned}$$

Since $x(0) = 0$ when the sponge initially has no salt water in it, we get:

$$0 = \frac{20}{39}(30) + \frac{C}{30^{8/5}} \implies C = \frac{20}{39}30^{13/5}.$$

Therefore the mass of salt is

$$x(t) = \frac{20}{39}(30 + \tfrac{5}{4}t) + \frac{20}{39}30^{13/5}\frac{1}{(30 + \frac{5}{4}t)^{8/5}}.$$

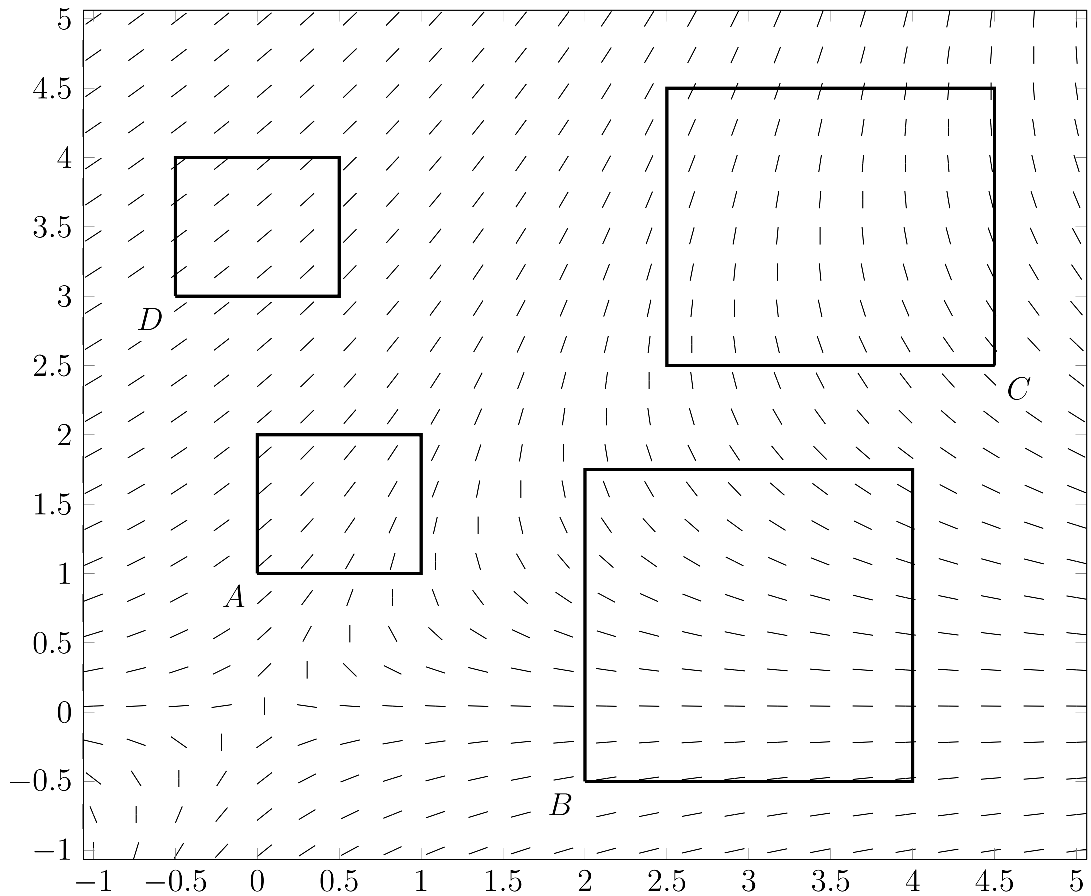
(c) (3 points) After a long time, you squeeze the sponge into a clean, dry container. How salty is the water?

Solution: The *salt concentration* in the sponge is $x(t)/(30 + \frac{5}{4}t)$:

$$\frac{20}{39} + \frac{20}{39}\left(\frac{30}{30 + \frac{5}{4}t}\right)^{13/5}.$$

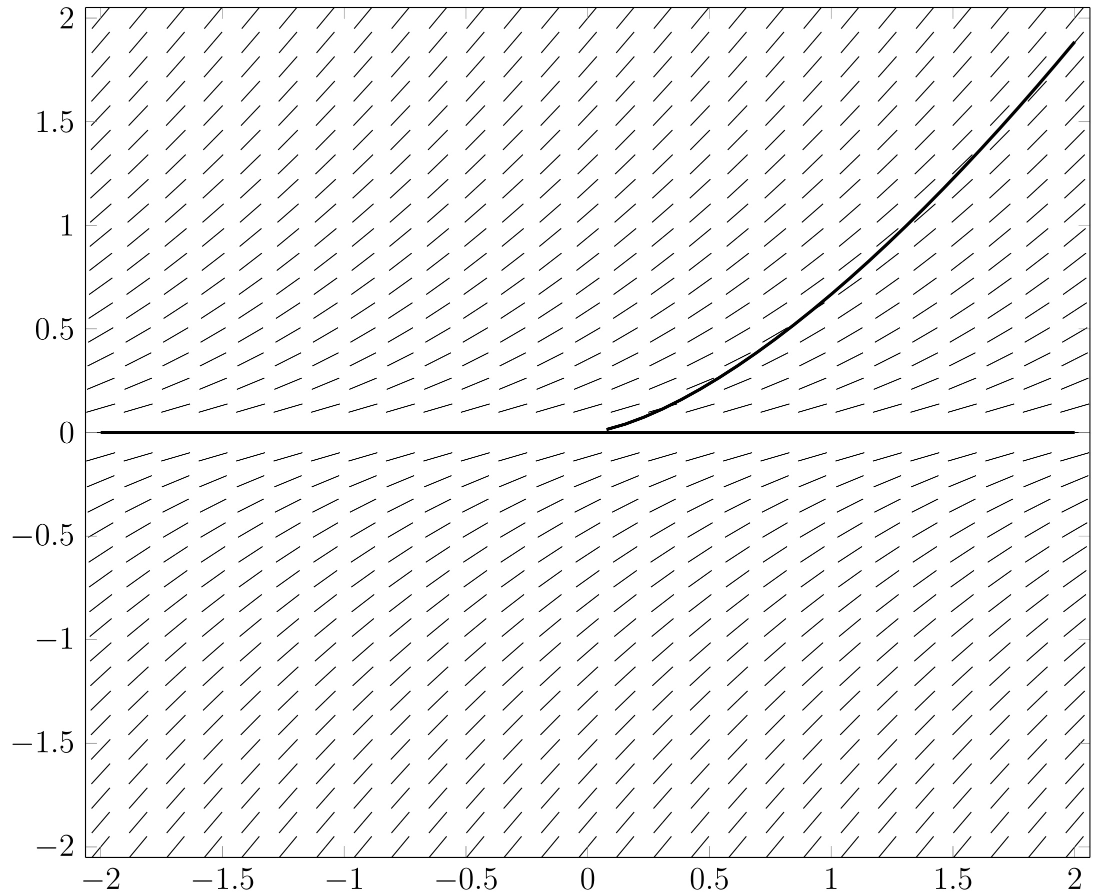
As $t \to \infty$, this approaches 20/39.

5. (a) (4 points) Below is a direction field for a certain differential equation $x'(t) = f(t, x)$ with several regions of the (t, x)-plane marked. In which region can we *not* expect a solution function $x(t)$ to exist? Explain why in the space provided. Sketch a solution curve in that region, emphasizing how it is not the graph of a function.



Solution: The answer is C, because the entire rectangle is crossed by a line of vertical slopes, indicating that $f(t, x)$ is discontinuous (in fact, infinite) there. The existence theorem therefore does not apply, so we cannot expect a solution function (there is a solution, not shown, that is not the graph of a function).

(b) (4 points) Below is a direction field for the differential equation $x'(t) = \sqrt{|x|}$; draw two different solution curves satisfying $x(0) = 0$, and explain in the space provided why this does not violate the uniqueness theorem.

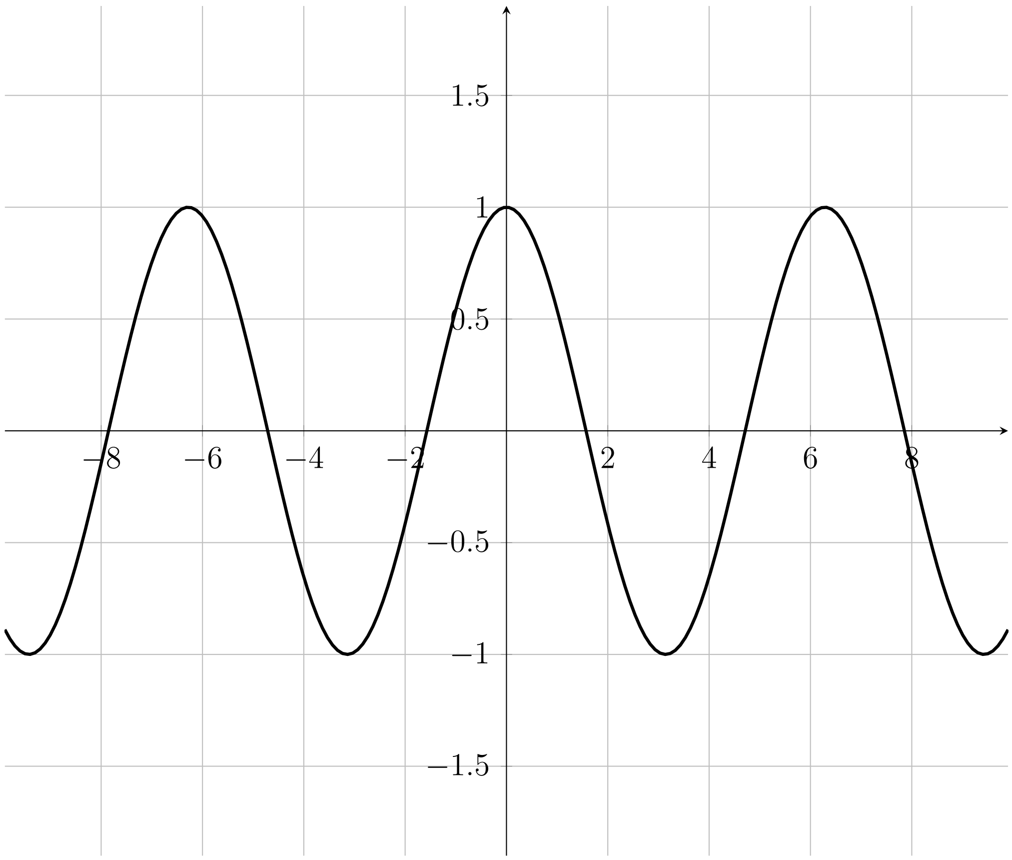


Solution: The uniqueness theorem requires $f(t,x) = \sqrt{|x|}$ and its partial x-derivative to be continuous, but we have

$$D_x\sqrt{|x|} = \begin{cases} 1/(2\sqrt{x}) & x > 0 \\ -1/(2\sqrt{-x}) & x < 0 \end{cases}$$

(it does not exist at $x = 0$) and therefore it is *not* continuous at $x = 0$, in fact infinite. Since the initial condition is $x(0) = 0$, the uniqueness theorem does not apply to this initial value problem and there may be multiple solutions.

(c) (4 points) Draw the phase line for the differential equation $x' = \cos(x)$ on the axes provided. In the space below, list (or give a formula for) all the equilibrium points, and state which are stable and which unstable. Explain your conclusions in the space provided.



Solution: The equilibria are the zeroes of $\cos(x)$, namely the numbers $\pi/2$, $3\pi/2, \ldots$ (that is, $n\pi/2$ with n odd). Stability can be tested by looking at the derivative, $-\sin(x)$, which is -1 at $\pi/2$, $5\pi/2, \ldots$ and 1 at the others. Therefore the numbers $n\pi/2$, with $n = 1, 5, 9, \ldots$ are stable equilibria and the others are unstable. The phase line is: