MATH 33B EXAM 1 Name:_

October 19, 2015 SID:_

Name of TA:_

I have read and understood the Student Honor Code, and this exam reflects my unwavering commitment to the principles of academic integrity and honesty expressed therein.

Signature ______________________

Each part of each problem is worth the number of points stated in parentheses. You must show all work to get any partial credit, which will be awarded for certain progress in a problem only if no substantially false statements have been written.

Please write your answers to problems A-E in the box below.

Problem	Answer	Points
1A	FALSE	2
1B	FALSE	2
1C	TRUE	2
1D	FALSE	2
1E	FALSE	2

10

This side for instructor's use only:

Problem	Points
2	10
3a	4
3b	4
3c	2
4	10

30

40

Problem 1. Below is a list of statements. Decide which are true and which are false. On the left of each, write "TRUE" or "FALSE" in capital letters. You must also write your answer ("TRUE" or "FALSE" in capital letters) on the front page of the exam.

There is no partial credit on this problem.

FALSE (A) (2 points) $y = e^t + e^{3t}$ is a solution to the differential equation $y'' - 4y' + 3 = 0$.

FALSE (B) (2 points) The differential form $\omega = xe^{xy}\,dx + ye^{xy}\,dy$ is exact everywhere.

TRUE (C) (2 points) The function $G(x, y) = \frac{x}{y} + \ln(x) - \ln(y)$ is homogeneous.

FALSE (D) (2 points) Solution curves to a differential equation never intersect.

FALSE (E) (2 points) The differential equation $(x')^2 = x$ has a unique solution satisfying the initial condition $x(0) = 0$.

A) $y' = e^t + 3e^{3t}$

$y'' = e^t + 9e^{3t}$

$e^t + 9e^{3t} - 4(e^t + 3e^{3t}) + 3 = 0$

$-3e^t - 3e^{3t} + 3 = 0$

$e^t - e^{3t} + 3 = 0$

B) $P = xe^{xy}$ $\quad Q = ye^{xy}$

$\partial P/\partial y = x^2e^{xy}$ $\quad \frac{\partial Q}{\partial x} = y^2e^{xy}$

C) $G(x, y, t) = \frac{x}{y} + \ln(tx) - \ln(ty)$

$= \frac{x}{y} + \ln\left(\frac{x}{y}\right)$

D) the differential eqn must be unique!

E) $\frac{dx}{dt} = x^{1/2}$

$\int x^{-1/2}\,dx$

$\frac{dx}{x^{1/2}} = dt$

$2x^{1/2}$

$2x^{1/2} = t + C_0$

$x^{1/2} = 2t + C$

$x = (2t + C)^2$

$x(0) = 0 = C^2$

$C = 0$. $\quad x(t) = 4t^2$

$f(t, x) = \pm\sqrt{x}$

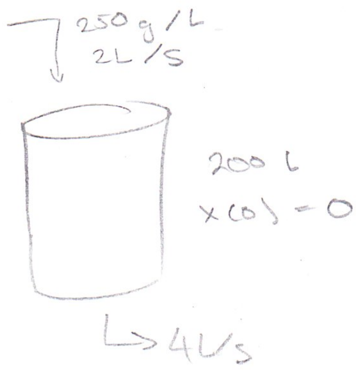
$\partial f/\partial x = \mp \frac{3}{2} x^{-1/2}$

10

Problem 2. (*10 points*) You must show all work to get partial credit.

There is a large tank initially filled with 200L of pure water. At time $t = 0$, a salt solution with a concentration of 250g/L is added to the tank at a rate of 2L/s. At the same time, a drain is opened which drains the contents at a rate of 4L/s. How many grams of salt are in the container when it is half full?

$$x' = \frac{250g}{L}\left(\frac{2L}{s}\right) - \frac{x\ g}{200-2t\ L}\left(4L/s\right)$$



$$x' = 500 - \frac{2x}{100-t}$$

$$x' + \frac{2x}{100-t} = 500$$

$$a(t) = -\frac{2}{100-t}$$

$$u(t) = e^{\int \frac{2}{100-t}\,dt} = e^{-2\ln|100-t|} = \frac{1}{(100-t)^2}$$

$$\left(\frac{x}{(100-t)^2}\right)' = \frac{500}{(100-t)^2} \qquad x^{-2} \qquad -x^{-1}$$

$$\frac{x}{(100-t)^2} = \frac{-500}{(100-t)} + C$$

$500 - 2C(100-t)$

$$x = -500(100-t) + C(100-t)^2$$

$$x(0) = 0 = -500(100) + C(100^2)$$

$$10{,}000\,C = 50000$$

25000
12500

$$C = 5$$

$$x(t) = -500(100-t) + 5(100-t)^2$$

Find t when tank is half full

$$200 - 2t = 100$$

$$-2t = -100$$

$t = 50$ seconds

$$x(50) = -500(50) + 5(50)^2$$

$$= 12{,}500 \text{ g salt}$$

Problem 3. (*10 points*) You must show all work to get partial credit.

Consider the differential equation
$$\frac{dy}{dx} = y(4-y)$$

(a) (4 points) Find the equilibria and determine whether they are stable or unstable.

Let $f(x,y) = y(4-y)$

find when $f(x,y) = 0 = (4-y)y$ $\rightarrow (5)$

$\rightarrow$ equilibria: 0 (unstable), 4 (stable)

4

(b) (4 points) Find the general solution. You must solve for y in terms of x.

$$\int \frac{dy}{y(4-y)} = \int dx$$

4

$$\int \frac{A}{y} + \frac{B}{4-y}\, dy = x + C_0$$

$A(4-y) + By = 1$

$4A = 1$

$-A + B = 0$

$A = 1/4$

$B = 1/4$

$$\frac{1}{4}\int \frac{1}{y} dy + \frac{1}{4}\int \frac{1}{4-y} dy = x + C_0$$

$$\frac{1}{4}\ln|y| - \frac{1}{4}\ln|4-y| = x + C_0$$

$$\ln\left|\frac{y}{4-y}\right| = 4x + C_1$$

$$\frac{y}{4-y} = Ce^{4x}$$

$$y = Ce^{4x}(4-y)$$

$$y(1 + Ce^{4x}) = 4Ce^{4x}$$

$$y(x) = \frac{4Ce^{4x}}{(1 + Ce^{4x})}$$

(c) (2 points) How many solutions satisfy $y(0) = 0$? What are they?

2

$$y(0) = 0 = \frac{4Ce^0}{1 + Ce^0} \Rightarrow C = 0$$

Uniqueness thm: $f(x,y) = y(4-y)$ continuous

$\longrightarrow$

$y(x) = 0$

$\partial f / \partial y = 4 - y - y = 4$ continuous

$\rightarrow$ Only 1 solution satisfies this initial condition

Problem 4. (*10 points*) You must show all work to get partial credit.

Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{1 - xy}{x^2 - xy}.$$

Your answer may be an implicit solution for y in terms of x.

$(x^2 - xy)\,dy + (xy - 1)\,dx = 0$

$(xy - 1)\,dx + (x^2 - xy)\,dy = 0$

$\partial P/\partial y = x \qquad \partial Q/\partial x = 2x - y$

try: $h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{x^2 - xy}\left(x - 2x + y\right)$

$= \frac{-x + y}{x(x-y)} = -\frac{1}{x}$

$u = e^{\int h(x)\,dx}$

$u = e^{-\ln|x|} = \frac{1}{x}$

$(y - \frac{1}{x})\,dx + (x - y)\,dy = 0$

$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 1$

$F(x,y) = xy - \ln|x| + \phi(y)$

$\partial F/\partial y = x + \phi'(y) = Q = x - y$

$\phi'(y) = -y$

$\phi(y) = -\frac{y^2}{2} + C$

$F(x,y) = xy - \ln|x| - y^2/2 = C$

10/10