MATH 33B PRACTICE EXAM 2

Problem 1. Verify that $y(t) = e^{\lambda t}$ and $y(t) = te^{\lambda t}$ are solutions to $y'' - 2\lambda y' + \lambda^2 y = 0$. Then prove they are linearly independent.

Problem 2. Find the general solution to $4y'' + y = 0$. Then find the particular solution that satisfies $y(1) = 0$ and $y'(1) = -2$.

Problem 3. A 100g = .1kg mass is hung from a spring having spring constant 9.8kg/s^2. The system is placed in a viscous medium that imparts a force of .3N when the mass is moving at .2m/s. Assume that the force applied to the medium is proportional, but opposite, to the mass' velocity. The mass is displaced 10cm from its equilibrium position and released from rest. Find the amplitude, frequency, and phase of the resulting motion.

Problem 4. Find the general solution to $y'' - 2y' + 5y = 3\cos t$. Then find the particular solution which satisfies $y(0) = 0$ and $y'(0) = -2$.

Problem 5. Use a guess of the form $y_p(t) = (at + b)e^{-4t}$ to find a particular solution to the equation $y'' + 3y' + 2y = te^{-4t}$.

Problem 6. Find the general solution to the equation $y'' - y' - 2y = 2e^{-t}$.

Problem 7. Find the characteristic polynomial, eigenvalues, and coresponding eigenvectors for

$$A = \begin{pmatrix} -2 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then find the general solution to $(\vec{x})' = A\vec{x}$.

Problem 8. Find the general solution of $(\vec{x})' = A\vec{x}$ where

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

Then sketch the half line solution, and sketch a rough approximation of a solution in each of the regions determined by the half-line solutions. Then find the particular solution when

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

What is the behavior of this solution as $t \to \infty$?

Problem 9. Consider the system $(\vec{x})' = A\vec{x}$ where

$$A = \begin{pmatrix} 4 & -3 \\ 15 & -8 \end{pmatrix}.$$

What kind of equilibrium point is $\vec{0}$? Compute the answer by finding the general solution to the differential equation, and then verify your answer by examining the corresponding point on the trace-determinant plane. Sketch a rough approximation of the solution curves.

Problem 10. Suppose x and y are two particular solutions to the second order linear differential equation

$$z''(t) + p(t)z'(t) + q(t)z(t) = f(t).$$

Prove that x and y differ by a solution to the corresponding homogeneous equation.

Problem 11. Suppose y is a solution to the second order linear differential equation with constant coefficients

$$y'' + 2cy' + \omega_0^2 y = 0 \qquad c > 0, \omega_0 > 0$$

Determine the behavior of the solution as $t \to \infty$. (There are 3 cases to consider.)

Problem 1.

Answer: Plug into the equation to check they are solutions. Compute the Wronskian to be $W(t) = e^{2\lambda t}$, which is clearly non-zero.

Problem 2.

Answer: $y(t) = A\sin(\frac{t}{2} - \phi)$, by solving characteristic equation and absorbing the second piece into a phase. initial value problem: $y(t) = 4\sin(\frac{1-t}{2})$.

Problem 3.

Answer: frequency is $\frac{\sqrt{167}}{2}$, amplitude is $\frac{7\sqrt{334}}{835}e^{-15t/2}$, phase is $\arctan(\frac{15}{\sqrt{167}})$.

Problem 4.

Answer: $y(t) = c_1 e^t \sin(2t) + c_2 e^t \cos(2t) - \frac{3\sin(t)}{10} + \frac{3\cos(t)}{5}$, initial value problem is $y(t) = \frac{1}{20}\left(-6\sin(t) - 11e^t\sin(2t) + 12\cos(t) - 12e^t\cos(2t)\right)$.

Problem 5.

Answer: $y_p(t) = e^{-4t}(\frac{5}{36} + \frac{t}{6})$.

Problem 6.

Answer: $y(t) = Ae^{-t} + Be^{2t} - \frac{2}{3}e^{-t}t$

Problem 7.

Answer: $-\lambda^3 + \lambda^2 + 4\lambda - 4$, $\lambda_1 = -2, v_1 = (1,0,0)$, $\lambda_2 = 2, v_2 = (5,4,0)$, $\lambda_3 = 1, v_3 = (14,9,-3)$. general solution is $y(t) = \sum_{i=1}^{3} C_i e^{\lambda_i t} v_i$.

Problem 8.

Answer: $y(t) = c_1 e^{2t} v_1 + c_2 e^{-t} v_2$ where $v_1 = (1,1)$ and $v_2 = (1,-2)$. use these to draw the half-lines. initial value, $c_1 = \frac{2}{3}$, $c_2 = \frac{1}{3}$. As $t \to \infty$, we approach the line spanned by v_1 from below.

Problem 9.

Answer: spiral sink.

Problem 10.

Answer: Define $G(t) = x(t) - y(t)$. plug $G(t)$ into the homogenous equation to check that it is a solution. after you plug in, rewrite $G(t)$ as $x - y$, then apply the fact that x, y solved the inhomogenous equation to just get $f(t) - f(t) = 0$.

Problem 11.

Answer: roots are $-c \pm \sqrt{c^2 - \omega_0^2}$.

case 1: overdamped. the solution is $y(t) = A_1 e^{(-c-\sqrt{c^2-\omega_0^2})t} + A_2 e^{(-c+\sqrt{c^2-\omega_0^2})t}$. Certainly $-c - \sqrt{c^2 - \omega_0^2}) < 0$ (negative number minus positive number). We also claim that $-c + \sqrt{c^2 - \omega_0^2} < 0$. To check this, note that $c^2 > c^2 - \omega_0^2$. Taking square roots and rearranging gives the desired inequality. Therefore, the solution is $y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$, where both r_1, r_2 are negative. Thus, $y(t) \to 0$ and $t \to \infty$.

case 2: underdamped. Let $\omega = \sqrt{\omega_0^2 - c^2}$. Then the solution is $y(t) = e^{-ct}(A_1 \cos(\omega t) + A_2 \sin(\omega t))$, which decays to zero as $t \to \infty$.

case 3: critically damped. The solution is $y(t) = A_1 e^{-ct} + A_2 t e^{-ct}$. As $t \to \infty$, $te^{-ct} \to 0$ by L'Hospital's rule. So, $y \to 0$ in this case as well.