MATH 33B EXAM 2 SOLUTIONS

Problem 1.

(A) (2 points) For which ϕ are the functions $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t+\phi)$ not linearly independent?
$\left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$
Solution: $\pi/2$ and $3\pi/2$. For these values of ϕ, $\sin(t+\phi) = \pm\cos(t)$, which is a scalar multiple of $\cos(t)$. The others give a multiple of $\sin(t)$, which is linearly independent from $\cos(t)$. This can be seen by computing a Wronskian.
Remark: The other version of the test had $\cos(t+\phi)$ and $\sin(t)$ instead, and the answer is the same.

(B) (2 points) Which of the following are solutions to the differential equation $y'' - 5y' + 4y = 0$?
$\{e^t, e^{2t}, e^{3t}, e^{4t}, e^{5t}, e^{6t}\}$
Solution: e^t and e^{4t}. This can be seen from factoring the characteristic polynomial $\lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1) = 0$.
Remark: The other version of the test had $y'' - 6y' + 5y = 0$, to which the answer is e^t and e^{5t}.

(C) (2 points) Let $p(t)$ and $q(t)$ be continuous on the interval (α, β). Suppose u and v are solutions to the differential equation $y'' + p(t)y' + q(t)y = 0$. For which $t \in (\alpha, \beta)$ can the Wronskian of u and v be zero?
{ all, some, none }
Solution: All or none. The Wronskian of two solutions is either always zero on the interval (this is the case that u and v are linearly dependent), or never zero on the interval (this is the case u and v are linearly independent).

(D) (2 points) If the differential equation $y'' + 2cy' + \omega_0^2 y = 0$ is overdamped, then the general solution involves terms of which type?
$\{e^{\lambda t}, te^{\lambda t}, \cos(\omega t), \sin(\omega t), e^{-ct}\cos(\omega t), e^{-ct}\sin(\omega t)\}$
Solution: $e^{\lambda t}$. The overdamped case is when there are two real roots to the characteristic polynomial.

(E) (2 points) Given the system $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}\mathbf{x}$, the origin is what type of equilibrium point?
{ saddle, nodal source, nodal sink, center, spiral source, spiral sink }
Solution: Center. Letting A denote the matrix, we compute $\text{Tr}(A) = 0$ and $\det(A) = -1 - 10 = 8$. This corresponds to a center point by looking at the trace determinant plane.

Remark: The other version of the exam had the matrix $B = \begin{pmatrix} 1 & 2 \\ 5 & -1 \end{pmatrix}$ which has trace 0 and determinant -12, corresponding to a saddle.

Problem 2. (*10 points*) Consider the second order differential equation $y'' - 2y' + y = e^t + \sin(t)$.

(a) (4 points) Find the general solution to the associated homogeneous differential equation.
Solution: We solve the characteristic equation $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$. This means the general homogeneous solution is $y_h(t) = C_1 e^t + C_2 t e^t$.

(b) (6 points) Find a particular solution, and use it to give the general solution.
Solution: We solve the two equations $y'' - 2y' + y = e^t$ and $y'' - 2y' + y = \sin(t)$ separately.

(1) We know that e^t and te^t are homogeneous solutions, so they will not work. Guessing $y_p(t) = At^2e^t$, we get

$$\begin{aligned} y_p'(t) &= 2Ate^t + At^2e^t = A(2t + t^2)e^t \\ y_p''(t) &= A(2 + 2t)e^t + A(2t + t^2)e^t = A(2 + 4t + t^2)e^t. \end{aligned}$$

Plugging in, we get the equation

$$y_p'' - 2y_p' + y_p = A(2 + 4t + t^2)e^t - 2A(2t + t^2)e^t + At^2e^t = 2Ae^t,$$

so $A = 1/2$.

(2) We guess $y_p(t) = A\cos(t) + B\sin(t)$. We compute

$$\begin{aligned} y_p'(t) &= -A\sin(t) + B\cos(t) \\ y_p''(t) &= -A\cos(t) - B\sin(t). \end{aligned}$$

Plugging in, we have

$$\begin{aligned} y_p'' - 2y_p' + y_p &= -A\cos(t) - B\sin(t) - 2(-A\sin(t) + B\cos(t)) + A\cos(t) + B\sin(t) \\ &= (-A - 2B + A)\cos(t) + (-B + 2A + B)\sin(t) \end{aligned}$$

Hence $B = 0$ and $A = 1/2$.

We now combine our answers to get the general solution:

$$y(t) = y_h(t) + \frac{1}{2}t^2e^t + \frac{1}{2}\cos(t) = C_1e^t + C_2te^t + \frac{1}{2}t^2e^t + \frac{1}{2}\cos(t).$$

Problem 3. (*10 points*) Consider the matrix $A = \begin{pmatrix} 2 & 2 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(a) (6 points) Find the general solution to the differential equation $\mathbf{x}' = A\mathbf{x}$.

Solution: One eigenvalue is easy to spot. If we subtract $2I$ from A, we get a row and column of zeroes, so $\det(A-2I) = 0$. Now we see that $(A-2I)e_3 = \vec{0}$, the third column of $A - 2I$, so $\vec{e_3}$ is an eigenvector corresponding to the eigenvalue 2, giving one solution

$$C_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now we only have to deal with the upper left 2×2 matrix. There, we see the characteristic polynomial $\lambda^2 - 6\lambda + 10$, which means the roots are

$$\lambda = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i.$$

Looking at $A - (3+i)I$, we have

$$\begin{pmatrix} -1-i & 2 & 0 \\ -1 & 1-i & 0 \\ 0 & 0 & -1-i \end{pmatrix}$$

and we observe that $\begin{pmatrix} 1-i \\ 1 \\ 0 \end{pmatrix}$ lies in the kernel. This gives us two more solutions

$$C_1 e^{3t} \left(\cos(t) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \sin(t) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right) + C_2 e^{3t} \left(\cos(t) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \sin(t) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right).$$

Adding these to the one above gives the general solution.

(b) (2 points) Find the particular solution when $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution: Plugging in $t = 0$ to the general solution, we get

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{x}(0) = C_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 - C_2 \\ C_1 \\ C_3 \end{pmatrix}$$

Hence $C_1 = C_3 = 1$ and $C_2 = 0$.

(c) (2 points) Describe the behavior of the particular solution as $t \to -\infty$.

Solution: The particular solution is

$$e^{3t} \left(\cos(t) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \sin(t) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right) + e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The behavior as $t \to -\infty$ is the same as the behavior as $t \to \infty$ of

$$e^{-3t}\left(\cos(-t)\begin{pmatrix}1\\1\\0\end{pmatrix} - \sin(-t)\begin{pmatrix}-1\\0\\0\end{pmatrix}\right) + e^{-2t}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= e^{-3t}\left(\cos(t)\begin{pmatrix}1\\1\\0\end{pmatrix} + \sin(t)\begin{pmatrix}-1\\0\\0\end{pmatrix}\right) + e^{-2t}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= e^{-2t}\left(e^{-t}\left(\cos(t)\begin{pmatrix}1\\1\\0\end{pmatrix} + \sin(t)\begin{pmatrix}-1\\0\\0\end{pmatrix}\right) + \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

Hence the particular solution is converging to $\vec{0}$ along the tangent $\begin{pmatrix}0\\0\\1\end{pmatrix}$. Moreover, if we look at the projection to the xy-plane, we see that we have the behavior of a spiral sink.

Problem 4. (*10 points*) Consider the case of critically damped harmonic motion

$$y'' + 2cy' + \omega_0^2 y = 0.$$

(a) (2 points) What are the conditions on c and ω_0 for the critically damped case?
Solution: The conditions for harmonic motion are $c \geq 0$ and $\omega_0 > 0$. Critically damped means that $c > 0$ and $\omega_0 = c$.

(b) (4 points) Prove that any solution decays to zero as $t \to \infty$.
Solution: The general solution for critically damped harmonic motion is

$$C_1 e^{-ct} + C_2 t e^{-ct} = e^{-ct}(C_1 + C_2 t).$$

Using L'Hospital's rule, we compute

$$\lim_{t\to\infty} e^{-ct}(C_1 + C_2 t) = \lim_{t\to\infty} \frac{C_1 + C_2 t}{e^{ct}} = \lim_{t\to\infty} \frac{C_2}{ce^{ct}} = 0.$$

(c) (4 points) Prove that any solution curve crosses the time axis at most once.
Solution: We want to know when a solution vanishes: $y(t) = e^{-ct}(C_1 + C_2 t) = 0$. This happens if and only if $C_1 + C_2 t = 0$. If $C_1 = C_2 = 0$, then the solution is always zero, so the solution curve *never* crosses the time axis. If $C_2 = 0$ and $C_1 \neq 0$, then the solution curve again never crosses the time axis, since $y(t) \neq 0$ for all t. If $C_2 \neq 0$, then $y(t) = 0$ exactly at $t = -C_1/C_2$, so the solution curve crosses the time axis exactly once.