1. Consider the differential equation

$$t^2y'' + 4ty' + 2y = 0 \qquad (\dagger)$$

(a) (2 points) Verify that $y_1(t) = \frac{1}{t}$ is a solution.

$y_1(t) = t^{-1}$

$y_1'(t) = -t^{-2}$

$y_1''(t) = 2t^{-3}$

$t^2y_1'' + 4ty_1' + 2y'$

$= t^2(2t^{-3}) + 4t(-t^{-2}) + 2t^{-1}$

$= t^{-1}(2-4+2) = 0$ ✓

✓

(b) (5 points) Suppose $y_2(t) = v(t)y_1(t)$ is another solution to the differential equation above. Use this to derive a second-order, linear differential equation for $v(t)$

$y_2(t) = v(t)y_1(t) = vt^{-1}$

$y_2'(t) = v'(t)y_1(t) + v(t)y_1'(t) = v't^{-1} - vt^{-2}$

$y_2''(t) = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) = v''t^{-1} - 2v't^{-2} + 2vt^{-3}$

$t^2y_2'' + 4ty_2' + 2y' = 0$

$t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 4t(v't^{-1} - vt^{-2}) + 2(vt^{-1}) = 0$

$v''t - 2v' + 2vt^{-1} + 4v' - 4vt^{-1} + 2vt^{-1} = 0$

$\boxed{v''t + 2v' = 0}$

✓

(c) (5 points) Solve the differential equation from part (b) and state the general solution to $(\dagger)$.

$v''t + 2v' = 0$

$v''t = -2v'$

$\frac{v''}{v'} = -\frac{2}{t}$

$\ln(v') = -2\ln t$

$v' = t^{-2}$

$v = \int \frac{1}{t^2}\,dt$

$v = -\frac{1}{t}$

$y_2(t) = vy_1 = -\frac{1}{t} \cdot \frac{1}{t} = -t^{-2}$

$y = C_1t^{-1} + C_2t^{-2}$ ✓

2. Consider the differential equation

$$y'' - 6y' + 8y = 40\cos(2t)$$

(a) (5 points) Find a fundamental set of solutions to the associated homogeneous equation.

$y(t) = e^{\lambda t}$ $y'(t) = \lambda e^{\lambda t}$ $y''(t) = \lambda^2 e^{\lambda t}$

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$e^{\lambda t}(\lambda^2 - 6\lambda + 8) = 0$

$(\lambda - 2)(\lambda - 4) = 0$

$\lambda = 2$ $\lambda = 4$

$\boxed{y_h = c_1 e^{2t} + c_2 e^{4t}}$

(b) (7 points) Find a particular solution to the inhomogeneous equation and then state the general solution.

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$y_p(t) = a\cos(2t) + b\sin(2t)$

$y'_p(t) = 2(-a\sin(2t) + b\cos(2t))$

$y''_p(t) = 4(-a\cos(2t) - b\sin(2t))$

$y''_p - 6y'_p + 8y_p = 4(-a\cos(2t) - b\sin(2t)) - 12(-a\sin(2t) + b\cos(2t)) + 8(a\cos(2t) + b\sin$

$= \cos 2t(-4a - 12b + 8a) + \sin(2t)(-4b + 12a + 8b)$

$= \cos 2t(4a - 12b) + (4b + 12a)\sin 2t = 40\cos 2t$

$4a - 12b = 40$

$4b + 12a = 0$

$12a = -4b$

$a = -\frac{1}{3}b$

$-\frac{4}{3}b - 12b = 40$

$-\frac{40}{3}b = 40$

$b = -3$ $a = 1$

$\boxed{y_p = \cos(2t) - 3\sin(2t)}$

2t

$\boxed{y = c_1 e^{2t} + c_2 e^{4t} + \cos(2t) - 3\sin(2t)}$

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3. Consider the differential equation

$$y'' - 8y' + 17y = 0$$

(a) (4 points) Determine the roots of the characteristic polynomial and state the associated (complex) solutions z_1, z_2.

$y(t) = e^{\lambda t}$ $\quad y'(t) = \lambda e^{\lambda t}$ $\quad y''(t) = \lambda^2 e^{\lambda t}$

$e^{\lambda t}(\lambda^2 - 8\lambda + 17) = 0$

$\lambda = \frac{8 \pm \sqrt{64-68}}{2} = 4 \pm i$

$$\boxed{\begin{array}{l} z_1(t) = e^{4t}(\cos(t) + i\sin(t)) \\ z_2(t) = e^{4t}(\cos(t) - i\sin(t)) \end{array}}$$

(b) (5 points) Explain why the real-valued functions $y_1(t) = \mathrm{Re}(z_1) = \frac{1}{2}(z_1 + z_2)$ and $y_2(t) = \mathrm{Im}(z_1) = \frac{1}{2i}(z_1 - z_2)$ are also solutions. Calculate $y_1(t)$ and $y_2(t)$ explicitly.

$y_1(t) = \frac{1}{2}(z_1 + z_2) = \frac{1}{2}e^{4t}(\cos(t) + i\sin(t) + \cos(t) - i\sin(t)) = \frac{1}{2}e^{4t}(2\cos t) = e^{4t}\cos(t)$

$y_2(t) = \frac{1}{2i}(z_1 - z_2) = \frac{1}{2i}e^{4t}(\cos(t) + i\sin(t) - \cos(t) + i\sin(t)) = \frac{1}{2i}e^{4t}(2i\sin t) = e^{4t}\sin t$

Since z_1 and z_2 are linearly independent solutions to the differential equation, any linear combination of z_1 and z_2 is also a solution. Therefore $y_1(t)$ and $y_2(t)$ are solutions to the diff eq as well

$$\boxed{\begin{array}{l} y_1(t) = e^{4t}\cos(t) \\ y_2(t) = e^{4t}\sin(t) \end{array}}$$

(c) (4 points) Determine the (real) solution with initial conditions $y(0) = 4$ and $y'(0) = -1$.

$y = C_1 e^{4t}\cos t + C_2 e^{4t}\sin t = e^{4t}(C_1\cos t + C_2\sin t)$

$y' = C_1(4e^{4t}\cos t - e^{4t}\sin t) + C_2(4e^{4t}\sin t + e^{4t}\cos t)$

$4 = C_1$

$-1 = 4C_1 + C_2$

$-1 = 16 + C_2$

$C_2 = -17$

$$\boxed{y = 4e^{4t}\cos t - 17e^{4t}\sin t}$$

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4. (13 points) Given that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2y'' + 3ty' - 3y = 0$$

Use variation of parameters to find the general solution to the inhomogeneous equation

$$t^2y'' + 3ty' - 3y = \frac{1}{t}$$

of the form $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$. (Remember: when calculating $y_p'(t)$, we set $v_1'y_1 + v_2'y_2 = 0$).

$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = \frac{1}{t^3}$

$yp(t) = v_1y_1 + v_2y_2$

$y'p(t) = \underbrace{v_1'y_1 + v_2'y_2}_{0} + v_1y_1' + v_2y_2'$

$y''p(t) = v_1'y_1' + v_2'y_2' + v_1y_1'' + v_2y_2''$

$y_1(t) = t \qquad y_2(t) = t^{-3}$

$y_1'(t) = 1 \qquad y_2'(t) = -3t^{-4}$

$W = \det\begin{pmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{pmatrix} = -3t^{-3} - t^{-3} = -4t^{-3}$

$v_1'y_1 + v_2'y_2 = 0$

$v_1'y_1' + v_2'y_2' = \frac{1}{t^3}$

$v_2' = \frac{y_1g}{W} = \frac{t\left(\frac{1}{t^3}\right)}{-4t^{-3}} = \frac{t^{-2}}{-4t^{-3}} = \frac{t}{-4} \qquad v_2 = \int -\frac{t}{4}\,dt = -\frac{t^2}{8}$

$v_1' = \frac{-y_2g}{W} = \frac{-t^{-3}\left(\frac{1}{t^3}\right)}{-4t^{-3}} = \frac{1}{4t^3} \qquad v_1 = \int \frac{1}{4t^3}\,dt = -\frac{1}{8t^2}$

$yp(t) = \left(-\frac{1}{8t^2}\right)(t) + \left(-\frac{t^2}{8}\right)(t^{-3}) = -\frac{1}{8t} - \frac{1}{8t} = \boxed{\frac{-1}{4t}}$

$\boxed{yp(t) = \frac{-1}{4t}}$

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