13 1. Consider the differential equation

$$x' = -x^2 - 2x + 3$$

(a) (5 points) Determine the equilibrium solutions. Classify each as stable or unstable

$f(x) = -x^2 - 2x + 3$

$f'(x) = -2x - 2$

5

$x' = -(x^2 + 2x - 3)$

$x' = -(x-1)(x+3)$

$x = 1$, $f'(1) < 0$, stable

$x = -3$, $f'(-3) > 0$, unstable

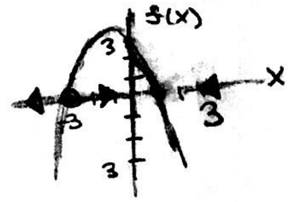
(b) (2 points) If x is a particular solution with initial condition $x(t_0) > 1$, what is $\lim_{t\to\infty} x(t)$?

2

~~$x(t_0) > 1$~~

$\boxed{\lim_{t\to\infty} x(t) = 1}$

$x = 1$ is a stable equilibrium point



(c) (6 points) Determine the particular solution with initial condition $x(0) = 2$.

6

$x' = -x^2 - 2x + 3$

$\frac{dx}{dt} = -(x-1)(x+3)$

$\frac{1}{(x-1)(x+3)} dx = -dt$

$\frac{1}{(x-1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+3)} = \frac{Ax + 3A + Bx - B}{(x-1)(x+3)} = \frac{(A+B)x + (3A-B)}{(x-1)(x+3)}$

$A + B = 0$ $A = -B$

$3A - B = 1$

$4A = 1$ $A = \frac{1}{4}$ $B = -\frac{1}{4}$

$\frac{1}{4}\int\left(\frac{1}{x-1} - \frac{1}{x+3}\right)dx = \int -dt$

$\frac{1}{4}(\ln|x-1| - \ln|x+3|) = -t + C$

$\ln\frac{|x-1|}{|x+3|} = -4t + C$

$\frac{x-1}{x+3} = Ce^{-4t}$

$x - 1 = (x+3)Ce^{-4t}$

$x(1 - Ce^{-4t}) = (3Ce^{-4t} + 1)$

$x = \frac{3Ce^{-4t} + 1}{1 - Ce^{-4t}}$

$x(0) = 2$

$2 = \frac{3C+1}{1-C}$

$2 - 2C = 3C + 1$

$5C = 1$

$C = \frac{1}{5}$

$x(t) = \frac{\frac{3}{5}e^{-4t} + 1}{1 - \frac{1}{5}e^{-4t}}$

$\boxed{x(t) = \frac{3e^{-4t} + 5}{5 - e^{-4t}}}$

$\lim_{t\to\infty} x(t) = \frac{5}{5} = 1$

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2. Blood carries a drug to an organ at a rate of 5 cm^3/sec and leaves at the same rate. The organ has a liquid volume of 125 cm^3. The concentration of the drug in the blood entering the organ is 0.2 mg/cm^3. Let $x(t)$ denote the amount of the drug in the organ at time t.

(a) (2 points) At what rate (in mg/sec) is the drug entering the organ?

rate in = volume in × concentration in = $\frac{5\ cm^3}{s} \times \frac{0.2\ mg}{cm^3} = \boxed{1\ mg/s}$

(b) (2 points) At what rate is the drug exiting the organ?

rate out = volume out × concentration out = $\frac{5cm^3}{s} \times \frac{x(t)\ mg}{125\ cm^3} = \boxed{\frac{x}{25}\ mg/s}$

(c) (5 points) Solve for $x(t)$ assuming that the person had no trace of the drug in their blood to start.

rate in - rate out = $1 - \frac{x}{25}$

$x' = 1 - \frac{x}{25}$

$\mu = e^{-\int -\frac{1}{25}dt} = e^{\int \frac{1}{25}dt} = e^{\frac{t}{25}}$

$(\mu x)' = 1(\mu)$

$\mu x = \int e^{\frac{t}{25}}dt$

$\mu x = 25e^{\frac{t}{25}} + C$

$x = \frac{1}{\mu}(25e^{\frac{t}{25}} + C)$

$x = 25 + Ce^{-\frac{t}{25}}$

$x(0) = 0$

$0 = 25 + C$

$C = -25$

$x(t) = 25 - 25e^{-\frac{t}{25}}$

$\boxed{x(t) = 25(1 - e^{-\frac{t}{25}})}$

(d) (4 points) The person will begin feeling the effect of the drug when the concentration in the organ is 0.1 mg/cm^3. How long after taking the drug will the person feel its effect? (leave your answer exact, don't worry about a decimal approximation.)

let t_f = time when drug is felt

concentration $= \frac{x}{V}$

$c = \frac{x}{V}$ $\quad x = Vc = (125cm^3)(0.1mg/cm^3) = 12.5\ mg$

$c(t_f) = 0.1\ mg/cm^3$

$x(t_f) = 0.1 \frac{mg}{cm^3} \cdot 125cm^3 = 12.5\ mg$

$12.5 = 25(1 - e^{-\frac{t_f}{25}})$

$\frac{1}{2} = 1 - e^{-\frac{t_f}{25}}$

$e^{-\frac{t_f}{25}} = \frac{1}{2}$

$-\frac{t_f}{25} = \ln(\frac{1}{2})$

$\boxed{t_f = 25\ln(2)\ \text{seconds}}$

3. Consider the differential equation

$$\frac{dx}{dt} = -(t + \cos t)x^2$$

(a) (5 points) Find the general solution.

$$\frac{dx}{dt} = -(t+\cos t)x^2$$

$$-\int \frac{1}{x^2}dx = \int (t+\cos t)dt$$

$$\frac{1}{x} = \frac{t^2}{2} + \sin t + C$$

$$\boxed{x(t) = \frac{1}{\frac{t^2}{2} + \sin t + C}}$$

✓

(b) (2 points) Determine the particular solution with initial condition $x(0) = 1$.

$x(0) = 1$

$$1 = \frac{1}{0 + c}$$

$$1 = \frac{1}{c} \qquad c = 1$$

$$\boxed{x(t) = \frac{1}{\frac{t^2}{2} + \sin t + 1}}$$

✓

(c) (3 points) What is the interval of existence to the solution in part (b)? Explain.

$$\frac{t^2}{2} + \sin t + 1 \neq 0$$

$$\frac{t^2}{2} + \sin t + 1 = 0 \qquad -\sin t = (\underbrace{\tfrac{t^2}{2}+1}_{\text{always} \geq 1})$$

$$\frac{t^2}{2} + \sin t = -1$$

$\boxed{\text{Interval of Existence } (-\infty, \infty)}$

$-1 \leq \sin t \leq 1$ & $\frac{1}{2}t^2 \geq 0$, so the sine function is the only possible way for our denominator to vanish

Only can vanish if $\sin(t) = -1$, $t = -\frac{\pi}{2}, \frac{3\pi}{2}$ But $\frac{1}{2}t^2 > 0$ for all t

sin t is always between -1 and 1

$\frac{t^2}{2}$ is always > 0

sin(t) would have to be less than -1 at some point for $\frac{t^2}{2} + \sin(t) + 1$ to be 0 so $\frac{t^2}{2} + \sin t + 1$'s never 0

✓

(d) (2 points) Determine the particular solution with initial condition $x(0) = 0$

~~$x(0) = 0$~~

~~$0 = \frac{1}{0 + c}$~~

~~no solution~~

-2

4. Consider the differential equation

$$y + (2y + kx)\frac{dy}{dx} = 0$$

(a) (4 points) What value of k makes the differential equation exact on the rectangle $(-\infty, \infty)$?

$y\,dx + (2y + kx)\,dy = 0$

$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = k$

IF EXACT $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, SO

if $k = 1$, the diff eq is exact

$\boxed{k = 1}$

(b) (5 points) Determine the general solution to the exact equation using the value of k found above.

$F(x,y) = \int P(x,y)\,dx + \phi(y) = \int y\,dx + \phi(y) = xy + \phi(y)$ 12/12

$Q(x,y) = \frac{\partial F}{\partial y} = x + \phi'(y)$

$Q(x,y) = 2y + x$

$2y + x = x + \phi'(y)$

$\phi'(y) = 2y$

$\phi(y) = y^2$

$\boxed{F(x,y) = xy + y^2 = c}$

(c) (3 points) Determine the particular solution with initial condition $y(1) = -2$ (you may leave your answer implicitly defined).

$y(1) = -2$

$(1)(-2) + (-2)^2 = c$

$-2 + 4 = c$

$c = 2$

$\boxed{xy + y^2 = 2}$

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