Solutions

Midterm 2

Spring 2015

Student information

First name: ______________ **Last name:** ______________

Student UID: ______________ **Discussion (circle one):** 2A| 2B | 2C | 2D | 2E | 2F

Course ID:	**MATH 33B**
Course title:	Differential equations
Instructor:	Aliki Mavromoustaki
Date:	11 May 2015
Exam duration:	50 minutes
Number of pages: (including this cover sheet)	11 pages
Exam Type:	**Closed Book**
Additional Materials Allowed:	**No**

Grade Table

Question	Score
1	/5
2	/5
3	/10
4	/15
5	/15
Σ	/50

1. (5 points) Given that $y = x$ is a solution of:

$$(x^2+1)y'' - 2xy' + 2y = 0, \qquad (1)$$

find a linearly independent solution by reduction of order.

Note: you may use appropriate formulas.

The eqn in SF is:

$$y'' \underbrace{- \frac{2x}{x^2+1}}_{P(x)} y' + \frac{2}{x^2+1} y = 0$$

$y_1 = x$ is a solution.

The second solution takes the form

$y_2 = v(x) y_1$ ① ($v(x)$: unknown)

Where $v(x) = \int \frac{1}{y_1^2} \cdot \exp\left[\int -P(x)\,dx\right] dx$. ②

$$e^{\int \frac{2x}{x^2+1} dx} = x^2+1 \quad ③$$

③ in ②: $v(x) = \int \frac{1}{x^2} \cdot x^2 + 1 \, dx$

$$v = x - \frac{1}{x}$$

In ①, $y_2 = x\left(x - \frac{1}{x}\right)$

Hence, a linearly indep. soln is:

$$\boxed{y_2 = x^2 - 1}$$

2. (5 points) The functions $y_1 = t^2$ and $y_2 = t^3$ are two distinct solutions of the initial value problem:

$$t^2 y'' - 4ty' + 6y = 0, \quad y(0) = 0, y'(0) = 0. \tag{2}$$

Why does this not violate the uniqueness theorem?

In SF, Eq. (2) is:

$$y'' - \frac{4t}{t^2} y' + \frac{6}{t^2} y = 0$$

with $P(t) = -\frac{4t}{t^2}$ & $Q(t) = \frac{6}{t^2}$.

for uniqueness, we need $P(t), Q(t)$ to be continuous over an interval containing the initial pt (here, $t_0 = 0$)

initial pt

Since P, Q are not cont. around $t=0$, the fact that $\exists$ 2 distinct solns to the IVP, doesn't violate the uniqueness thm.

3. (10 points) Consider the following differential equation for logistic population growth with harvesting:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \lambda N \tag{3}$$

where r, K and λ are positive constants. For the following cases, compute the critical points, determine their stability and sketch the equilibrium and some non-equilibrium solutions.

(a) $\lambda < r$;

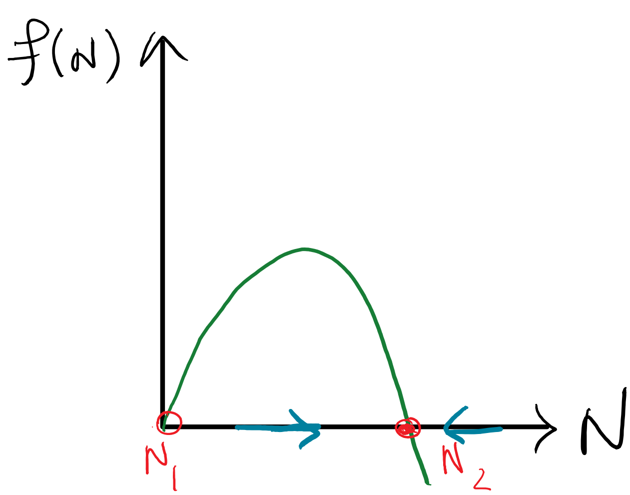
(b) $\lambda = r$.

Critical pts occur at $N' = 0$.

$$\Rightarrow N\left[r\left(1 - \frac{N}{K}\right) - \lambda\right] = 0$$

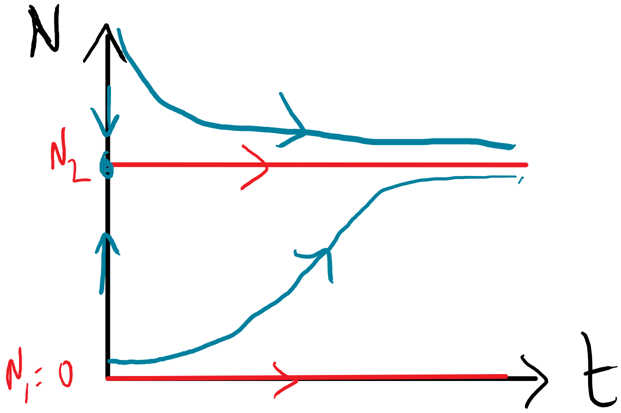
$$\therefore N_1 = 0 \text{ \& } N_2 = K\left(1 - \frac{\lambda}{r}\right)$$

(a) Stability



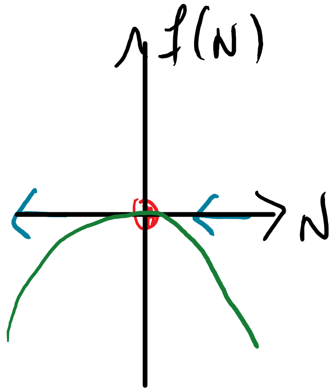
$\therefore N_1 = 0$ is unstable while N_2 is asymptotically stable

Note: $N \geq 0$ for physical significance

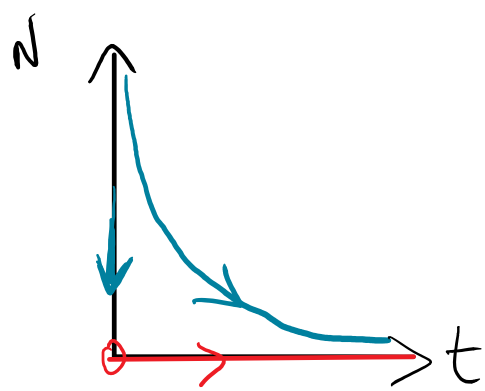


(b) for $\lambda = r$, $N_2 = 0$ hence $\exists$ 1 c.p. at 0

Stability



The c.p is semi-stable



4. (a) Find the general solution of

$$y^{iv} - 4y''' + 14y'' - 20y' + 25y = 0,$$

if the corresponding characteristic equation has the following roots

$$1 + 2i,\ 1 - 2i,\ 1 + 2i,\ 1 - 2i.$$

Note that $y = y(x)$ and y^{iv} represents the fourth order derivative with respect to x.

(b) Using the method of undetermined coefficients, set up the correct form for a particular solution y_p to the following nonhomogeneous differential equation:

$$y^{iv} - 4y''' + 14y'' - 20y' + 25y = e^{2x}\sin(2x) + e^{x}\cos(2x). \qquad (4)$$

Note: do not solve for the undetermined coefficients.

(a) 4th order LODE. The roots of the characteristic polynomial give:

$m_1 = 1+2i,\ m_2 = 1-2i,\ m_3 = 1+2i,\ m_4 = 1-2i$

Since m_3, m_4 are repeated, we multiply the corresponding solns by x:

$$y_h = e^{x}(c_1 \sin 2x + c_2 \cos 2x) + xe^{x}(c_3 \sin 2x + c_4 \cos 2x)$$

where $c_1 - c_4$ are arbitrary real constants.

(b) $f(x) = \underbrace{e^{2x}\sin 2x}_{f_1} + \underbrace{e^{x}\cos 2x}_{f_2}$

Guess for f_1: $y_{p_1} = e^{2x}(A\sin 2x + B\cos 2x)$

Guess for f_2: $y_{p_2} = e^{x}(C\sin 2x + D\cos 2x)$ ← multiply by x^2 to make y_p L.I. to solns contained in y_h

$$y_p = e^{2x}(A\sin 2x + B\cos 2x) + e^{x}x^2(C\sin 2x + D\cos 2x)$$

5. (a) Using the method of variation of parameters from first principles, **show** that the equation $y'' + y = f(x)$ leads to the particular solution:

$$y_p(x) = \int_0^x f(t)\sin(x-t)\,dt.$$

(b) Find a similar formula for a particular solution of the equation $y'' + k^2 y = f(x)$ where k is a positive constant.

(a) $y'' + y = f(x)$ ① has as a particular soln of the form: $y_p = v_1(x)y_1 + v_2(x)y_2$ ② where v_1, v_2 are unknown nonconstant functions.

$v_1(x), v_2(x)$ satisfy: $\boxed{y_1 v_1' + y_2 v_2' = 0}$ ③ (necessary assumption)

Using ③, y_p' is: $y_p' = v_1 y_1' + v_2 y_2'$ ④

and y_p'': $y_p'' = v_1 y_1'' + y_1' v_1' + v_2 y_2'' + y_2' v_2'$ ⑤

Sub. ②, ④ & ⑤ in ① and rearranging:

$v_1(\underbrace{y_1'' + y_1}_{=0}) + v_2(\underbrace{y_2'' + y_2}_{=0}) + y_1' v_1' + y_2' v_2' = f(x)$ ⑥

since y_1, y_2 are solns to the homog. ODE.

∴ ⑥ reduces to: $\boxed{y_1' v_1' + y_2' v_2' = f(x)}$ ⑦

③ & ⑦ are 2 indep. eqns for v_1', v_2'.

from ③, $v_2' = -\frac{v_1' y_1}{y_2}$ ⑧

In ⑦: $y_1' v_1' + y_2' \left(\frac{-v_1' y_1}{y_2}\right) = f(x)$

$$v_1' \left(\frac{y_1' y_2 - y_1 y_2'}{y_2}\right) = f(x) \Rightarrow v_1' = \frac{f(x)\, y_2}{y_1' y_2 - y_1 y_2'}$$

⑨ $v_1 = -\int \frac{y_2 f(x)}{W(x)} dx$

From ⑧, $v_2 = \int \frac{y_1 f(x)}{W(x)} dx$ ⑩

Homog. soln:

$y'' + y = 0 \Rightarrow$ char. eqn: $m^2 + 1 = 0$

$$y_h = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$$

$W(y_1, y_2) = 1$. In ⑨, ⑩

$$v_1 = -\int \sin x\, f(x)\, dx \quad \text{and} \quad v_2 = \int \cos x\, f(x)\, dx$$

where note that constants of integration are ignored WLOG s.t. we can write:

$$\boxed{v_1 = \int_0^x -\sin t\, f(t)\, dt \quad \text{and} \quad v_2 = \int_0^x \cos t\, f(t)\, dt}$$

t here is used as the dummy variable

Using the boxed eqns above in y_p:

$$y_p = \int_0^x f(t)\left[-\sin t \cos x + \cos t \sin x\right] dt$$

Using the sine difference formula:

$$y_p = \int_0^x f(t)\left[\sin(x-t)\right] dt$$ as required.

(b) $y'' + k^2 y = f(x)$.

The homog. soln is: $y_h = c_1 \cos kx + c_2 \sin kx$

$W(y_1, y_2) = k$

Hence,

$$v_1 = \frac{1}{k}\int_0^x -\sin kt\, f(t)\, dt \quad \text{and} \quad v_2 = \int_0^x \cos kt\, f(t)\, dt$$

Therefore, y_p is:

$$\boxed{y_p = \frac{1}{k}\int_0^x f(t)\left[\sin k(x-t)\right] dt}$$

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