Math 33B: Differential Equations

Midterm 1

Wednesday, Oct. 29, 2014 9:00 - 9:50 AM *Instructor: aliki*

Name: ____

PLEASE PRINT

UID: __________

Section: ________

Discussion sections:

- *Tuesday* with: X. Luo - **1A** | B. West - **1C**
- *Thursday* with: X. Luo - **1B** | B. West - **1D**

Read the following information before starting the exam:

- Show **all** your work, clearly and in order;
- This test has **5 questions** and is worth a total of **50** points;
- No books, notes, electronic devices (inc. calculators) are allowed;
- Good luck!! ☺

QUESTION #	SCORE	MAX. POINTS
1		10
2		10
3		10
4		10
5		10
TOTAL		**50**

Question 1 (*10 points*)

Solve the initial value problem:

$$\frac{dy}{dx} = y + 2xe^x, \quad y(0) = 3.$$

BLANK PAGE

Question 2 (*10 points*)

Solve the following ODE for an equation of the form $f(x, y) = c$:

$$\frac{dy}{dx} = \frac{(2x\cos y + 3x^2y)}{(y + x^2\sin y - x^3)}.$$

Apply the initial condition $y(0) = 2$ to find a particular solution in *implicit* form.

BLANK PAGE

Question 3 (*10 points*)

Consider the following ODE,

$$2xy\frac{dy}{dx} = y^2 - x^2.$$

By making an appropriate change of variables, transform the ODE into a separable one. Hence, find an implicit solution for $y(x)$.

BLANK PAGE

Question 4 (*10 points*)

Consider the following ODE:

$$\frac{dy}{dt} = y^2. \tag{1}$$

(a) Draw a direction field for the ODE given by Eq. (1).

(b) Solve the ODE to find a general solution for $y(t)$ (in explicit form).

(c) Find a particular solution for the initial condition $y(0) = 1$. What is the interval of existence?

(d) Find a particular solution $y(t)$, for the initial condition $y(0) = y_0$ (where y_0 is a constant). There exists a finite time T such that

$$\lim_{t \to T} y(t) = \infty.$$

What is this time T?

BLANK PAGE

Question 5 (*10 points*)

A simple model for the spread of an infection in a population is given by:

$$\begin{aligned} \frac{dH}{dt} &= -kIH, \\ \frac{dI}{dt} &= kIH, \end{aligned} \tag{2}$$

where $H(t)$ represents the amount of healthy people, $I(t)$ represents the amount of infected people and k is the rate of infection. Since,

$$\frac{d}{dt}(H+I) = 0, \tag{3}$$

it follows that the size of the population, $H+I$ is equal to a constant value, say N:

$$H + I = N. \tag{4}$$

(a) Using the substitution $I = N - H$, obtain a single equation for $\frac{dH}{dt}$.

(b) Find the critical points of the ODE you obtained in (a). Determine their stability (i.e. stable, unstable or semi-stable).

(c) Sketch the equilibrium and some non-equilibrium solutions in the $H - t$ plane.

BLANK PAGE

Q.1

I don't have time to write up the solutions neatly. These were my rough, very quick solutions when setting the exam and so the existence of silly mistakes is very likely. I hope these are still useful though. The boxed numbers represent points awarded at each step but it may be that we didn't go by this grading scheme in the end - so ignore the boxed numbers! :)

$$y' - y = 2x e^{x}.$$

$$\text{I.F.} \quad \mu = e^{\int -1\,dx} = e^{-x} \qquad \boxed{3}$$

$$e^{-x} y' - e^{-x} y = 2x.$$

$$\frac{d}{dx}\left[e^{-x} y\right] = 2x \qquad \boxed{6}$$

$$e^{-x} y = x^2 + C \Rightarrow y = e^{x} x^2 + Ce^{x}.$$

At $x = 0$, $y(0) = C = 3$ $\qquad \boxed{8}$

$$\Rightarrow y(x) = e^{x}\left(x^2 + 3\right)$$

$$\boxed{10}$$

Q.2

Rewrite: $\left(y+x^2\sin y - x^3\right)\frac{dy}{dx} - \left(2x\cos y + 3x^2 y\right) = 0.$

$P(x,y) = -\left(2x\cos y + 3x^2 y\right) \qquad Q(x,y) = y + x^2\sin y - x^3.$

$\frac{\partial P}{\partial y} = -\left(-2x\sin y + 3x^2\right) \qquad \frac{\partial Q}{\partial x} = 2x\sin y - 3x^2$

$P_y = Q_x \Rightarrow$ exact. — 3

Find $f(x,y)$ s.t. $\frac{d}{dx}\left[f(x,y)\right] = P(x,y) + Q(x,y)\frac{dy}{dx} = 0$

$\frac{\partial f}{\partial x} = P \Rightarrow f = -\int 2x\cos y + 3x^2 y \, dx = -\left(x^2\cos y + x^3 y\right) + c_1(y)$

$\frac{\partial f}{\partial y} = Q \Rightarrow f = \int y + x^2\sin y - x^3 \, dy = \frac{y^2}{2} - x^2\cos y - x^3 y + c_2(x)$ — 6

$\Rightarrow f = -x^2\cos y - x^3 y + \frac{y^2}{2}$ — 7

where $c_1(y) = y^2/2$, $c_2(x) = 0$

G.S: $x^2\cos y + x^3 y - \frac{y^2}{2} = C$ — 9

Apply I.C: $-2 = C \Rightarrow x^2\cos y + x^3 y - \frac{y^2}{2} = -2$

Q.5

$$2xy\frac{dy}{dx} = y^2 - x^2.$$

Rewrite: $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{y}{2x} - \frac{x}{2y}$. *

let $u = y/x$ ⇒ $y = u \cdot x$ where $u = u(x)$.

[2]

$y' = u + x\frac{du}{dx}$. — [4]

In *

$$u + x\frac{du}{dx} = \frac{1}{2}u - \frac{1}{2u}$$

$$x\frac{du}{dx} = -\left(\frac{u}{2} + \frac{1}{2u}\right) = -\left(\frac{u^2+1}{2u}\right)$$ — [7]

$$\int \frac{2u}{u^2+1}\,du = \int -\frac{1}{x}\,dx.$$

by observation or substitution,

$$\ln(u^2+1) = -\ln x + \ln c.$$

$$u^2 + 1 = \frac{c}{x} \Rightarrow u^2 = \frac{c}{x} - 1$$

but $u = y/x$

$$\boxed{y^2 = cx - x^2}$$ — [10]

Q.4

$y' = y^2$

(a) $\boxed{3}$

(b) $\int \frac{1}{y^2}\,dy = \int dt \Rightarrow -\frac{1}{y} = t + c$

$y = \frac{-1}{t+c}.$ — 5

(c) $y(0)=1$, $y(0) = \frac{-1}{c} = 1 \Rightarrow c = -1$

$\boxed{y(t) = \frac{1}{1-t}}$ — 6

$t \neq 1$

$\boxed{t < 1}$ — 7

↑ includes I.C. ($t=0$)

(d) $y(0) = y_0$. $y(0) = \frac{-1}{c} = y_0 \Rightarrow c = -1/y_0$

$y(t) = \frac{-1}{t - 1/y_0} = \frac{y_0}{1 - y_0 t}$ — 8

As $t \to 1/y_0$ $y(t) \to \pm\infty$

— 10

Q. 6

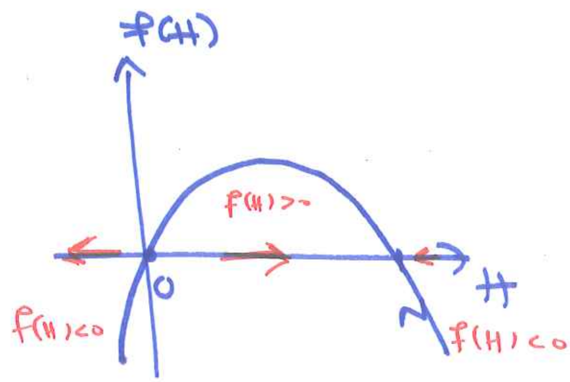
$$\frac{dH}{dt} = -kIH \qquad \frac{dI}{dt} = kIH.$$

(a) $$\boxed{\frac{dH}{dt} = -k(N-H)H}$$

2

(b) $\underbrace{-k(N-H)H}_{f(H)} = 0$ $H = N$ or $H = 0$.

4

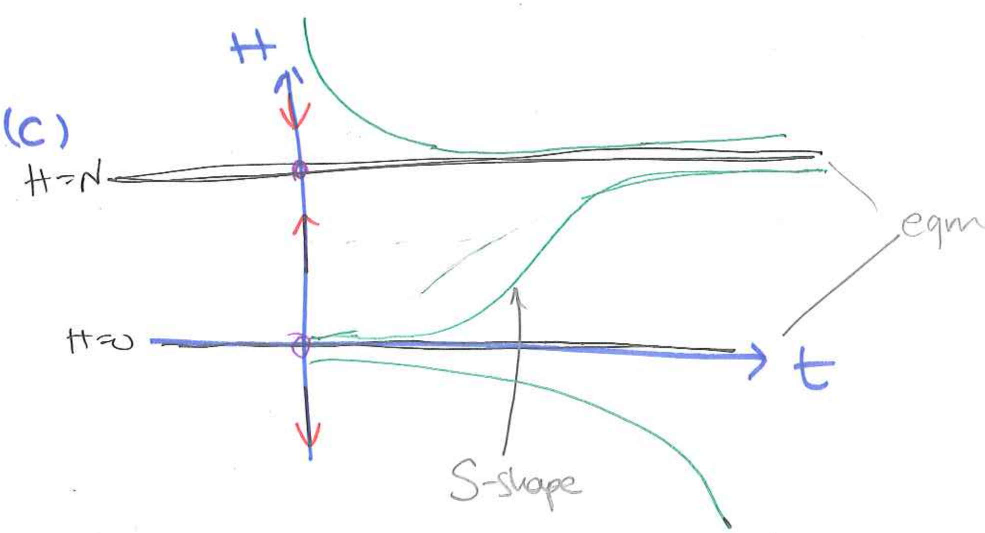


H=0 is ←o→ unstable

H=N is →•← stable.

7

(c)



10