MATH 33B - 1
Differential Equations – Midterm 1
Version 1A

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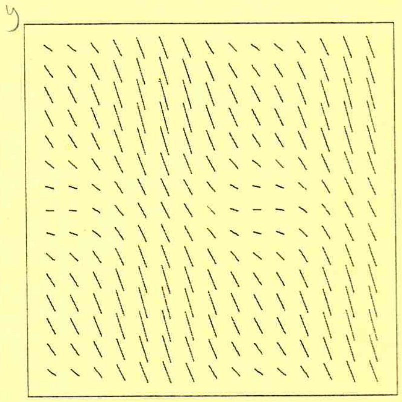
First Name:	
Last Name:	
Bruin ID:	
TA Name:	
Section:	
Sect. day:	

Directions: This test is to be completed **without** the use of notes, books, or technology. The time limit is 50 minutes. You must show all your work to get full credit, except for the multiple choice questions (Q1 and Q2).

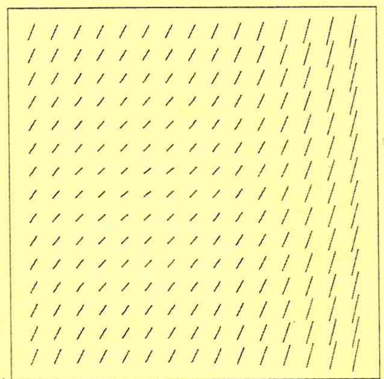
(For grading purposes only – Leave blank)

Q1	10	6
Q2	10	8
Q3	10	12
Q4	10	10
Q5	12	12
Q6	6	6
Total	58	54

Question 1. (10 pts) Please match the following slope fields with their associated differential equations. Circle the correct answers.



(a)



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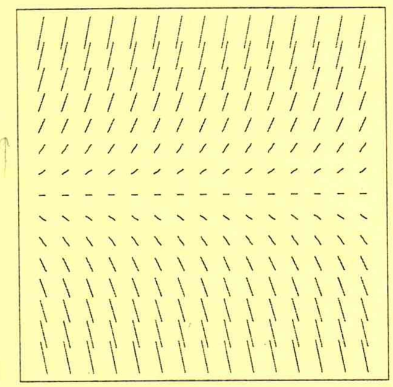
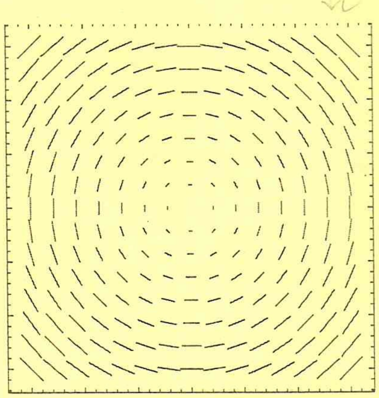
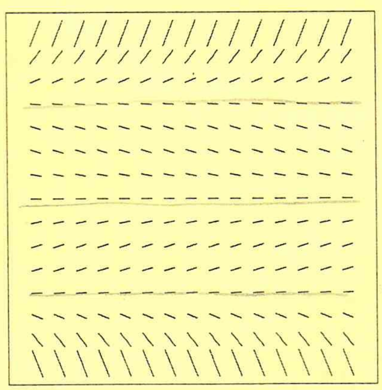


(c)



(d)



(e)

1) $y' = \frac{-x}{y}$ a b c d e b

0,3,-3 2) $y' = y(y-3)(y+3)$ a b c d e e a,b,d

3) $y' = y^2 + x^2 + e^x$ a b c d e d b,d

0 or always negative 4) $y' = \sin x + \cos y - 2$ a b c d e a

5) $y' = y$ a b c d e c

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Question 2. (10 pts) Which of the following differential 1-forms ω are exact on the domain $\mathbb{R}^2 \setminus (0,0)$? Circle the correct answer.

a) $\omega := dF$, for $F(x,y) = 2x^2 + y^2$ Exact Not-exact

b) $\omega := dF$, for $F(x,y) = \frac{x-y}{x^2+y^2}$ Exact Not-exact

c) $\omega := (x+y)dx + (x-y)dy$ Exact Not-exact

d) $\omega := xdx + (y-x)dy$ Exact Not-exact

e) $\omega := \frac{y}{x^2+y^2}dx + \frac{-x}{x^2+y^2}dy$ Exact Not-exact

a) $F(x,y) = 2x^2+y^2$

$dF = 4x\,dx + 2y\,dy$

$0 = 0$ Exact.

b) $dF = \cancel{\frac{x-y}{(x+y)(x-y)}}$

$F = (x-y)(x^2+y^2)^{-1}$

$\frac{dF}{\partial x} = (x-y)\cdot -(x^2+y^2)^{-2}\cdot 2x + (x^2+y^2)^{-1}$

$P = -\frac{(x-y)2x}{(x^2+y^2)^2} + \frac{1}{(x^2+y^2)}$

$\frac{\partial F}{\partial y} = -\frac{(x-y)2x}{(x^2+y^2)^2} - \frac{1}{(x^2+y^2)} = Q$

$\frac{\partial P}{\partial y} = (\text{some}) + -\frac{2y}{(x^2+y^2)}$

$\frac{\partial Q}{\partial x} = (\text{some}) + \frac{2x}{(x^2+y^2)}$ } $\to$ not equal.

2xy

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

c) $1 = 1$ ✓

d) $\frac{\partial(x)}{\partial y} = \frac{\partial(y-x)}{\partial x}$

$0 \neq -1$

e) $\frac{1}{x^2+y^2} \cdot \frac{-y\cdot 2y}{(x^2+y^2)^{-2}}$

$= \frac{-1}{x^2+y^2} \cdot \frac{+x\cdot 2x}{(x^2+y^2)^2}$

$-2y^2 \neq -2x^2$

8

Question 3. Consider the initial value problem

$$y' = \sin y, \quad y(0) = 3.$$

Remember: when using any theorem from class, make sure to explain why it applies.

a) (2 pts) Does there exist a solution $y(t)$ to this initial value problem? Why?

We use the theorem in this way

If y' is cts on some rect. R, and some initial condition (t_0, y_0) is on R, there exists a solution. This applies because if we know that a valid slope can be associated with every point and form a continuous line, we can definitely define some solution very close to that point. (ε)

$y' = \sin y$ is defined on $t \in \mathbb{R}$ and $y \in \mathbb{R}$, (continuous everywhere),
so there exists a solution.

$\mathbb{R}$ = all reals

+2

b) (2 pts) If so, how many solutions exist? Why?

We can check uniqueness:

t & $y \in \mathbb{R}$ = all reals

$y'' = \cos y$. This is cts on the same interval, so this guarantees us a $\boxed{\text{unique}}$ solution. (Only 1 solution exists.)

Our theorem states this because the second derivative's cts requires that no solution curves overlap. Since another curve (solution) cannot ~~pass~~ intersect this point, it must be unique. If the second derivative is continuous, we see that ~~the~~ we can draw only non-intersecting solutions.

+2

4

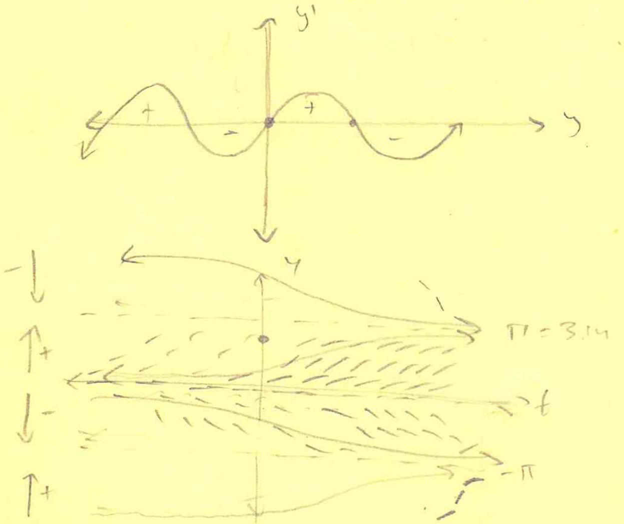
c) (4 pts) If $y(t)$ is a solution to the initial value problem,

$$y' = \sin y, \qquad y(0) = 3,$$

what is $\lim_{t\to\infty} y(t)$? Justify your response.

$y(0)=3$

$y(1)=3\ldots?$



If we consider this as an autonomous function, we find that we can produce the following slope diagram. As you can see, there is an equilibrium point at $y=0, \pi, -\pi$. $n\pi$

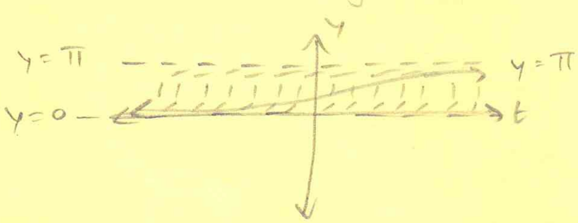
If we consider $t=0$; $y=3$ in this plot, we see that the form of the solution carries it to the upper asymptote at π.

So, $\lim_{t\to\infty} y(t) = \pi$.

+4

d) (4 pts) If $y(t)$ is a solution to the above initial value problem, then for which value(s) of t does $y(t) = -1$? Justify your answer.

There is no value of t such that $y(t) = -1$. Referring back to the diagram:



If $y(t)$ is a solution containing $y(0)=3$, then it is clear that $y(t)$ is bounded by the two equilibrium points 0 and π.

Since $0 < y(t) < \pi$, there can be no value of t such that $y(t) = -1$. +4

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Question 4. (10 pts) Solve the initial value problem:

$$y' = y + 2xe^{2x}, \quad y(0) = 3.$$

$y' = a(x)y + 2xe^{2x}$ $a(x) = 1$
$b(x) = 2xe^{2x}$

$\mu(x) = e^{-\int a(x)dx} = e^{-x}$

$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)b(x)\,dx + C\right]$

$y(x) = \frac{1}{e^{-x}}\left[\int e^{-x}\cdot 2xe^{2x}\,dx + C\right]$

$y(x) = e^{x}\left[\int 2xe^{x}\,dx + C\right]$

$= e^{x}\left[2\int xe^{x}\,dx + C\right]$

$= e^{x}\left[2e^{x}(x-1) + C\right]$

$y(x) = 2e^{2x}(x-1) + Ce^{x}$

Integral:

$\int xe^{x}dv$

$u = x$ $dv = e^{x}dv$

$du = dx$ $v = e^{x}$

$xe^{x} - \int e^{x}dv$

$xe^{x} - e^{x}$

$e^{x}(x-1)$

$y(0) = 3 = 2e^{0}(0-1) + Ce^{0}$

$3 = -2 + C$ $C = 5$

$$\boxed{y(x) = 2e^{2x}(x-1) + 5e^{x}}$$

✓

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Question 5. (12 pts) Solve the following initial value problem for an equation of the form $F(x, y) = C$ that defines y *implicitly* as a function of x:

$$y' = \frac{(2x\cos y + 3x^2 y)}{(y + x^2 \sin y - x^3)}, \quad y(0) = 2.$$

$$\frac{dy}{dx} = \frac{(2x\cos y + 3x^2 y)}{(y + x^2 \sin y - x^3)}$$

$$(y + x^2 \sin y - x^3)\,dy = (2x\cos y + 3x^2 y)\,dx$$

$$(-2x\cos y - 3x^2 y)\,dx + (y + x^2 \sin y - x^3)\,dy = 0$$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 2x\sin y - 3x^2 - (2x\sin y - 3x^2) = 0 \leftarrow \text{EXACT}$$

$$\int P\,dx = \int -2x\cos y - 3x^2 y\,dx = \underbrace{-x^2\cos y - x^3 y}\, + \phi(y)$$

$$\int Q\,dy = \int y + x^2 \sin y - x^3\,dy = \frac{1}{2}y^2 \underbrace{- x^2 \cos y - x^3 y}\, + \phi(x)$$

General Solution $\boxed{F(x,y) = \frac{1}{2}y^2 - x^2\cos y - x^3 y = C}$

$$F(0,2) = \frac{1}{2}(4) - 0\cos(2) - 0(2) = C$$
$$= 2 = C$$

Solution $y(0)=2$ $\boxed{F(x,y) = \frac{1}{2}y^2 - x^2\cos y - x^3 y = 2}$

+12

Question 6. (6 pts) The differential form

$$\omega := \overset{P}{2y}dx + \overset{Q}{(x+y)}dy$$

is not exact. Find an integration factor of the form $\mu(x,y) = \mu(y)$ so that the product $\mu(y)\cdot\omega$ is equal to dF for some continuously differentiable function $F(x,y)$ on the domain $(0,\infty)\times(0,\infty)$. Note: you **do not** need to find this function F, just find $\mu(y)$.

$$\mu(y) = e^{-\int g(y)\,dy}$$

$P = 2y \quad Q = x+y$

$\frac{\partial P}{\partial y} = 2 \quad \frac{\partial Q}{\partial x} = 1$

where $g(y) = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$g(y) = \frac{1}{2y}(2-1) = \frac{1}{2y}$$

$$\mu(y) = e^{-\int \frac{1}{2y}dy} = e^{-\frac{1}{2}\ln y} = (e^{\ln y})^{-1/2} = y^{-1/2} = \boxed{\frac{1}{\sqrt{y}}}$$

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