K.T.

1. (20 points) Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y' = x/(1+2y), \quad y(-1) = 0.$$

$$y' = \frac{x}{1+2y}$$

$$\frac{dy}{dx} = \frac{x}{1+2y}$$

$$dy = \left(\frac{x}{1+2y}\right)dx$$

$$(1+2y)\,dy = (x)\,dx$$

$$\int(1+2y)\,dy = \int x\,dx$$

$$y + y^2 = x^2 + C$$

$$y^2 + y = x^2 + C$$

Taking into account the initial value $y(-1)=0$:

$$y^2 + y\Big|_{y=0} = \frac{x^2}{2} + C\Big|_{x=-1}$$

$$0 + 0 = \frac{(-1)^2}{2} + C$$

$$0 = \frac{1}{2} + C$$

$$C = \frac{-1}{2}$$

$$\boxed{y^2 + y = \frac{x^2}{2} - \frac{1}{2}}$$

+10

Int. of existence:
"$y \in (-\infty, -\frac{1}{2})$"
$x \in (-\infty, \infty)$

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2. (20 points) Find the solution of the initial value problem.

$$(1+t^2)y' + 4ty = (1+t^2)^{-2}, \quad y(1) = 0.$$

$$\frac{(1+t^2)y'}{(1+t^2)} + \frac{4ty}{(1+t^2)} = \frac{(1+t^2)^{-2}}{(1+t^2)}$$

$$y' + \left(\frac{4t}{1+t^2}\right)y = \frac{1}{(1+t^2)^3}$$

$$y' = -\left(\frac{4t}{1+t^2}\right)y + \frac{1}{(1+t^2)^3}$$

Using variation of parameters:

$$y = y_h \cdot v$$

$$y_h' = \left(-\frac{4t}{1+t^2}\right)y$$

$$\frac{dy}{dt} = \left(\frac{-4t}{1+t^2}\right)y$$

$$\frac{1}{y}dy = \left(\frac{-4t}{1+t^2}\right)dt$$

$$\int \frac{1}{y}dy = -2\int\left(\frac{2t}{1+t^2}\right)dt$$

$$\ln|y| = -2\ln|1+t^2|$$

$$e^{\ln|y|} = e^{-2\ln|1+t^2|}$$

$$y_h = e^{\ln\left|\frac{1}{(1+t^2)^2}\right|} = \frac{1}{(1+t^2)^2}$$ ✓ +7

$$\Downarrow$$

$$v' = \frac{f}{y_h} = \frac{\frac{1}{(1+t^2)^3}}{\frac{1}{(1+t^2)^2}} = \frac{(1+t^2)^2}{(1+t^2)^3} = \frac{1}{1+t^2}$$

$$v = \int \frac{1}{1+t^2}dt = \tan^{-1}(t) + C$$

✓ +7

$$y = y_h \cdot v = \frac{1}{(1+t^2)^2}\cdot\left(\tan^{-1}(t) + C\right)$$

$$y = \frac{\tan^{-1}(t)}{(1+t^2)^2} + \frac{C}{(1+t^2)^2}$$

Initial value $y(1)=0$:

$$0 = \frac{\tan^{-1}(1)}{(1+1)^2} + \frac{C}{(1+1)^2}$$

$$= \frac{\frac{\pi}{4}}{(2)^2} + \frac{C}{(2)^2}$$

$$0 = \frac{\pi}{16} + \frac{C}{4} \qquad C = -\frac{\pi}{4}$$ ✓ +6

$$\boxed{y = \frac{\tan^{-1}(t)}{(1+t^2)^2} - \frac{\pi}{4(1+t^2)^2}}$$ ✓

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K.T.

3. (20 points) Find the integrating factor to make the following equation into an exact equation. Then find the general solution. (If you remember the integrating factor, you can use it directly.)

$$(x^2y^2 - 1)ydx + (1 + x^2y^2)xdy = 0.$$

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If μ is the integrating factor for the differential form above, then

$\mu = \frac{1}{xy}$.

Multiplying μ into the diff. form above gives us the equation:

$$x\left(xy - \frac{1}{xy}\right)y\,dx + x\left(\frac{1}{xy} + xy\right)dy = 0$$

$$\underbrace{\left(xy^2 - \frac{1}{x}\right)}_{P}dx + \underbrace{\left(\frac{1}{y} + x^2y\right)}_{Q}dy = 0$$ ✓

$$\frac{\partial P}{\partial y} = 2xy \qquad \frac{\partial Q}{\partial x} = 2xy$$ ✓

As $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, the equation is exact and can be solved.

$$F(x,y) = \int P\,dx + \phi(y) = \int\left(xy^2 - \frac{1}{x}\right)dx + \phi(y)$$

$$= \frac{x^2y^2}{2} - \ln|x| + \phi(y)$$ ✓

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^2y^2}{2} - \ln|x| + \phi(y)\right] = \frac{\cancel{2}x^2y}{\cancel{2}} + \phi'(y) = x^2y + \phi'(y) = Q$$

$$\cancel{x^2y} + \phi'(y) = \cancel{x^2y} + \frac{1}{y}$$ ✓

$$\phi'(y) = \frac{1}{y} \rightarrow \phi(y) = \ln|y| + C$$

$$\boxed{F(x,y) = \frac{x^2y^2}{2} - \ln|x| + \ln|y| = C}$$ ✓ good!

K.T.

4. (20 points) Suppose that x is a solution to the initial value problem

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$x' = x - t^2 + 2t$ $\qquad$ $\cancel{x' = \frac{x^3 - x}{1 + t^2x^2}}$, $\quad x(0) = 1.$

Show that $x(t) > t^2$ for all t for which x is defined.

$$x' = f(t,x) = x - t^2 + 2t$$

- f is continuous for all t and x, and is thus continuous on the entire tx-plane. ✓
- $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x - t^2 + 2t] = 1 - 0 + 0 = 1.$ ✓
  - $\frac{\partial f}{\partial x}$ is continuous for all t and x, and is thus continuous on the entire tx-plane.

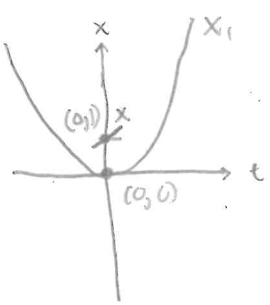
$\therefore$ As f and $\frac{\partial f}{\partial x}$ are continuous on the entire tx-plane, which includes the point $(t_0, x_0) = (0,1)$, there is a unique solution to the IVP above by uniqueness theorem. ✓

But, x' has another solution x_1 such that

$x_1(t) = t^2$:

$x_1' = x_1 - t^2 + 2t$

$2t = t^2 - t^2 + 2t$ $\quad\longrightarrow\quad$ $2t = 2t$ ✓ ✓



But, we see that $x_1(0) = 0$ and $x(0) = 1$. ✓

Therefore, by the graph to the left and the observation above, we see that x cannot intersect x_1 anywhere on the tx-plane, so as to uphold the uniqueness theorem.

Hence: $x(t) > x_1(t)$

or

$\therefore$ $x(t) > t^2$ for all t on the tx-plane ✓ good!

K.T.

5. (20 points) Find the general solution for the following differential equation.

$$4y'' + 4y' + y = 0.$$

Using the characteristic equation, we see that the equation above becomes:

$$4\lambda^2 + 4\lambda + 1 = 0$$

$$(2\lambda+1)(2\lambda+1) = 0 \rightarrow (2\lambda+1)^2 = 0$$

$$\lambda = -\frac{1}{2}$$

By using property of characteristic roots, we see that $\lambda = -\frac{1}{2}$ is a real, repeated root, so the solutions (linearly independent) for the 2nd order linear, homogenous ODE with constant coefficients (the equation above) are:

$$y_1(t) = e^{\lambda t} = e^{-t/2}$$

$$y_2(t) = te^{-t/2}$$

$\therefore$ The general solution is:

$$\boxed{y(t) = C_1 e^{-t/2} + C_2 te^{-t/2}}$$