1. (25 points) The isotope Iodine 131 is used to destroy tissue in an overactive thyroid gland. It has a half-life of 8.04 days. If a hospital receives a shipment of 600 mg of Iodine 131, how much of the isotope will be left after 30 days?

25

$N = N_0 e^{-\lambda t}$

$N(0) = 600$

$N_0 = 600$ at $t = 0$.

b/c $N = N_0 e^0$

$N_{1/2} = 300$

$\frac{1}{2} = e^{-\lambda(8.04)}$

$-\frac{\ln(\frac{1}{2})}{8.04} = \lambda$

$\lambda = 0.0862$

$\frac{N}{N_0} = \frac{1}{2} = e^{-\lambda t}$

$-\frac{\ln|\frac{1}{2}|}{t_{1/2}} = \lambda$

t in days

$N = N_0 e^{-0.0862t}$

$600 = N_0$

so

$N = 600 e^{-0.086(30)}$

$\boxed{N = 45.17 \text{ mg Iodine 131 after 30 days}}$

2. (25 points) A 100-gal tank initially contains 40 gal of pure water. Sugar-water solution containing 2 lb of sugar for each gallon of water begins entering the tank at a rate of 4 gal/min. **After 10 minutes**, a drain is opened at the bottom of the tank, allowing the sugar-water solution to leave the tank at a rate of 2 gal/min. What is the sugar content (lb) in the tank at the precise moment that the tank is full of sugar-water solution?

Before 10 minutes

rate in = $2\frac{lb}{gal} \times 4\frac{gal}{min} = 8\frac{lb}{min}$

$V = 40 + 4(10) = 80$

$X(10) = ? \Rightarrow 80$

$X(0) = 0$

rate out = $0\frac{gal}{min} \times \frac{x(t)}{40+4t} = 0$

$x'(t) = 8 - 0 \Rightarrow x = 8t_1,\ t_1 = 10,\ x = 80$ lb sugar

$x(10) = 80$ lb

After 10 minutes

(After 10 mins) $80 + (4-2)t$

Volume $= 80 + 2t_2$, $t_2 = 10$ when volume = 100 (capacity)

rate in $= 8\frac{lb}{min}$ rate out = $2\frac{gal}{min} \times \frac{x(t)}{80+2t} = \frac{x(t)}{40+t}$

assume t is t_2 in scope of after 10 min

$x'(t) = 8 - \frac{x(t)}{40+t}$ $\quad x'(t) + \frac{x(t)}{40+t} = 8$

not integrating factor $\rightarrow a(t) = \frac{1}{40+t} \Rightarrow u(t) = e^{\int \frac{1}{40+t}dt} = e^{\ln|40+t|}$

$u(t) = 40+t$

in T_2 perspective

$x(0) = 80$

$\int [x(t)(40+t)]' = \int (40+t)\, 8\, dt$

$x(t)(40+t) = 4(40+t)^2 + C$

t_1

$x(0) = 80$ lb from previous t in new scope of volume

$x(t) = 4(40+t) + \frac{C}{40+t}$

$80 = 4(40+0) + \frac{C}{40+0}$

$80 = 160 + \frac{C}{40}$ $\quad C = -3200$

$x(t) = 4(40+t) - \frac{3200}{40+t}$

$x(10) = 4(50) - \frac{3200}{50}$

$\boxed{x(10) = 136 \text{ lb sugar}}$

when tank is full

3. (25 points) Solve the following differential equation:

$$(y^2 - xy)dx + (xy - 1)dy = 0$$

$\frac{d}{dy}(y^2 - xy) = 2y - x \quad \frac{d}{dx}(xy-1) = y$

$\frac{dP}{dy} = 2y - x \qquad 2y - x \neq y$ (not exact) $\frac{dQ}{dx} = y$

$\mu = \mu(y) \Rightarrow \frac{1}{P}\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) \to g = \frac{1}{y^2 - xy}(2y - x - y)$

$= \frac{1}{y^2 - xy}(y - x)$

$\frac{1}{y(\cancel{y-x})}\cancel{(y-x)}$

$\frac{1}{y} = g$

$\mu(y) = e^{-\int \frac{1}{y} dy} = e^{-\ln|y|} = \frac{1}{y} = \mu(y)$

$\frac{1}{y}(y^2 - xy)dx + \frac{1}{y}(xy - 1)dy = 0$

$(y - x)dx + (x - \frac{1}{y})dy = 0$

$\frac{d}{dy}(y - x) = 1 \qquad \frac{d}{dx}(x - \frac{1}{y}) = 1$

$1 = 1$, exact after $\mu(y)$ multiplies.

So,

$F(x,y) = \int (y - x)dx = xy - \frac{x^2}{2} + \phi(y)$

$\frac{dF}{dy} = \frac{d}{dy}\left(xy - \frac{x^2}{2} + \phi(y)\right) = Q(x,y)$

$\Rightarrow \cancel{x} - 0 + \phi'(y) = (\cancel{x} - \frac{1}{y})$

$\phi(y) = \int -\frac{1}{y} dy$

$= -\ln|y| + C$

Great Job!

25/25

So,

$\boxed{F(x,y) = xy - \frac{x^2}{2} - \ln|y| = C}$

$\frac{dF}{dx} = y - \frac{2x}{2} = y - x$ ✓

$\frac{dF}{dy} = x - 0 - \frac{1}{y} = x - \frac{1}{y}$ ✓

4. (25 points) Solve the following differential equation:

$$(2xe^{\frac{y}{x}} - y)dx + xdy = 0$$

this is homogeneous of degree 1 $\Rightarrow$ let $y = vx$

$dy = vdx + xdv$

$\frac{y}{x} = \frac{vx}{x} = v$

$(2xe^{v} - vx)dx + x(vdx + xdv) = 0$

$= 2xe^{v}dx - \cancel{vxdx} + \cancel{xvdx} + x^2dv$

$= 2xe^{v}dx + x^2dv$

$= \int \frac{2}{x}dx + \int e^{-v}dv = 0$

$v = y/x$

$= 2\ln|x| + -e^{-v} + C = 0$

$2\ln|x| - e^{-\frac{y}{x}} + C = 0$

$e^{-y/x} = 2\ln|x| + C$

$-\frac{y}{x} = \ln(C + 2\ln|x|)$

$\boxed{y = -x\ln(C + 2\ln|x|)}$

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