Midterm 1

Last Name: ____________________

First Name: ____________________

Student ID: ____________________

Signature: ____________________

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Khang Huynh
	1C	1D	TA: Eli Sadovnik
	1E	1F	TA: Jason Snyder

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Question	Points	Score
1	15	14
2	15	15
3	10	5
4	10	10
Total:	50	44

√1. (a) (5 points) Find the solution y_h to the differential equation:

$$y' = \frac{1}{x}y$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|y| = \ln|x| + C$$

$$\boxed{y_h = Cx}$$

(b) (10 points) Solve the initial value problem:

$$y' = \frac{1}{x}y + \sqrt{x}, y(1) = 0$$

have to divide whole eqn by x -1pt

IF: $\mu(x) = e^{\int -\frac{1}{x}dx}$

$= e^{-\ln|x|}$

$= \frac{1}{x}$

$$-\frac{1}{x^2}y + \frac{1}{x}y' = \sqrt{x}/x$$

$$\left(\frac{1}{x}y\right)' = \sqrt{x}/x$$

$$\frac{y}{x} = \int x^{1/2}\,dx$$

$$\frac{y}{x} = \frac{2}{3}x^{3/2} + C$$

$$\frac{0}{1} = \frac{2}{3}(1)^{3/2} + C$$

$$0 = \frac{2}{3} + C$$

$$C = -\frac{2}{3}$$

$$\boxed{y = \frac{2}{3}x^{5/2} - \frac{2}{3}x}$$

2. (a) (5 points) Find the general solution $y_h = C_1y_1 + C_2y_2$ to the differential equation:

$$y'' + y = 0$$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$ $\quad(\alpha \pm \beta i)$

$\boxed{y_h = C_1 \cos t + C_2 \sin t}$ /

✓

(b) (10 points) Use undetermined coefficient or variation of parameters, find the general solution to the differential equations

$$y'' + y = t + e^t$$

$y_p = at + ke^t$

$y' = a + ke^t$

$y'' = ke^t$

$ke^t + at + ke^t = t + e^t$

$2ke^t + at = t + e^t$

$a = 1$

$2k = 1$

$k = \frac{1}{2}$

$y_p = t + \frac{1}{2}e^t$

$y = y_p + y_h$

$\boxed{y = t + \frac{1}{2}e^t + C_1 \cos t + C_2 \sin t}$ ✓

3. (10 points) Solve the homogeneous equation:

$$(y^2 + 2xy)dx - x^2 dy = 0$$

(Hint: Using $y = vx$ change the differential equation to a seperable equation)

$dy = v\,dx + x\,dv$

$(v^2x^2 + 2vx^2)\,dx - x^2(v\,dx + x\,dv) = 0$

$x^2(v^2 + 2v)\,dx - x^2(v\,dx + x\,dv) = 0$

$v^2\,dx + 2v\,dx - v\,dx - x\,dv = 0$

$v^2\,dx + v\,dx - x\,dv = 0$

$(v^2 + v)\,dx = x\,dv$

$\frac{1}{v} + \frac{1}{v+1} = \frac{v+1+v}{v(v+1)} = \frac{2v+1}{v(v+1)}$

$\int \frac{1}{x}\,dx = \int \frac{1}{v^2+v}\,dv$

$\ln|x| = \int \frac{1}{v^2+v}\,dv$ AND?

(Plug $v = \frac{y}{x}$, solve for y)

-5

Last six digits of UID:

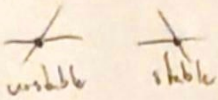
4. Consider the autonomous equation:

$$y' = y(y-2)e^y$$

(a) (2 points) Find the equilibrium solutions of the above differential equations.

For $y \equiv 0$ and $y \equiv 2$, $y' = 0$, making the soln. y a constant and at equilibrium

(b) (3 points) Determine the stability of the equilibrium solutions.



$y' = y^2e^y - 2ye^y$

$y'' = \cancel{2ye^y} + y^2e^y - 2e^y - \cancel{2ye^y}$

$= y^2e^y - 2e^y$

$y''(0) = 0^2e^0 - 2e^0 = -2 < 0$ ∴ stable @ $y = 0$

$y''(2) = 4e^2 - 2e^2 = 2e^2 > 0$ ∴ unstable @ $y = 2$

(c) (5 points) Prove that if $y(t)$ is a solution and $y(0) = 1$, then $0 < y(t) < 2$ for all $t \in (-\infty, \infty)$.

$0 < y(t)$ Let $y(t_1) < 0$ for some $t_1 \in \mathbb{R}$. By the IVT, there must exist some $t_0 \in (t_1, 0)$ or $(0, t_1)$ such that $y(t_0) = 0$. Looking at the equations for y' and y'' above, one can see that both are continuous and defined, ∴ the theorem of uniqueness and existence applies. Above we found that $y \equiv 0$ is an equilibrium solution of the DE. Letting $g(t) \equiv 0$ be a solution of the DE and taking t_0 as the initial condition for both $y(t)$ and $g(t)$, we find that $y(t_0) = 0$ and $g(t_0) = 0$. This violates the uniqueness theorem, ∴ $y(t) > 0$ for $t \in \mathbb{R}$ by contradiction.

$y(t) < 2$ Similarly, let $y(t_2) > 2$ for $t_2 \in \mathbb{R}$. IVT shows that there exists a t_0 where $y(t) = 2$. Let $h(t) \equiv 2$ be a solution to the DE. @ t_0, $y(t_0) = h(t_0) = 2$, so by uniqueness thm $y = h$. However, $y(t)$ also satisfies $y(0) = 1$, while $h(0) = 2$, so $y \neq h$. This contradicts the uniqueness thm, ∴ $y(t) < 2$ for all $t \in \mathbb{R}$.