Midterm 1

Last Name:

First Name:

Student ID:

Signature:

Section:

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Question	Points	Score
1	15	14
2	15	13
3	10	10
4	10	10
Total:	50	47

Fall

1. (a) (5 points) Find the solution y_h to the differential equation:

$$y' = \frac{1}{x}y$$

homogeneous eq

$\frac{dy}{dx} = \frac{1}{x}y$

$\frac{1}{y}dy = \frac{1}{x}dx$

$\ln|y| = \ln|x| + c$

$e \quad e$

$y = e^c \cancel{e^{\ln|x|}} = Ax$

$y = \boxed{y_h = x}$

what happened to the abs. values - 1pt.

can get rid of constant cuz it's still a homogenous sol.

You should keep the const in.

(b) (10 points) Solve the initial value problem:

$$y' = \frac{1}{x}y + \sqrt{x}, y(1) = 0$$

$y' = \frac{1}{x}y + \sqrt{x} \longrightarrow f$

$y_h = x$ finding an equation v such that $y = vy_h$

we know $v' = \frac{f}{y_h} \longrightarrow \int dv = \int \frac{\sqrt{x}}{x}dx \longrightarrow v = \int (x)^{\frac{1}{2}}(x)^{-1}dx$

some constant c

$v = \int (x)^{-\frac{1}{2}}dx = \left(\frac{2x^{\frac{1}{2}}}{1} + c\right) = 2\sqrt{x} + c$

so general $y = Ax(2\sqrt{x} + c)$

$y(1) = 0 = 2(1)^{3/2} + 1c$

$0 = 2 + c$

$-2 = c$

so $\boxed{y = x(2\sqrt{x} - 2)}$

assuming y(t)

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2. (a) (5 points) Find the general solution $y_h = C_1y_1 + C_2y_2$ to the differential equation:

$$y'' + y = 0$$

substituting in $y = e^{\lambda t}$

$\lambda^2 + 1 = 0$

$(\lambda + i)(\lambda - i)$

$\lambda^2 - \cancel{\lambda i} + \cancel{\lambda i} - i^2$

$\lambda^2 - i^2$

$\lambda^2 + 1$

$\lambda = \pm i$

These are imaginary with $\lambda = a \pm bi = 0 \pm 1i$

$\boxed{y_h = c_1 \cos(t) + c_2 \sin(t)}$ ✓

(b) (10 points) Use undetermined coefficient or variation of parameters, find the general solution to the differential equations

$$y'' + y = t + e^t$$

Break into 2 → undet coeff.

$y'' + y = t$

$y_1 = at + b$

$y_1' = a$

$y_1'' = 0$

$0 + at + b = t$

$b = 0 \quad a = 1$

so $y_1 = t$

$y'' + y = e^t$

$y_2 = ae^t$

$y_2' = ae^t \quad y_2'' = ae^t$

$ae^t + ae^t = e^t$

$2ae^t = e^t$

$a = \frac{1}{2}$

so $y_2 = \frac{1}{2}e^t$

so $\boxed{y_p = t + \frac{1}{2}e^t}$ + homog s/n

checking more...

$y_p' = 1 + \frac{1}{2}e^t$

$y_p'' = \frac{1}{2}e^t$

−2

so

$\frac{1}{2}e^t + t + \frac{1}{2}e^t = t + e^t$

✓ $t + e^t = t + e^t$ ✓

3. (10 points) Solve the homogeneous equation:

$$(y^2 + 2xy)dx - x^2 dy = 0$$

(Hint: Using $y = vx$ change the differential equation to a seperable equation)

This is homogeneous of the same degree.

so $y = vx$.

$x^2(xdv + vdx) = x^3dv + x^2vdx$

$((vx)^2 + 2x\,vx)dx - x^2\,d(vx) = 0$

$(v^2x^2 + 2x^2v - x^2v)dx - x^3dv = 0$

divide by x^2

$(v^2 + 2v - v)\,dx = x\,dv$

$\int \frac{1}{x}\,dx = \int \frac{1}{v^2+v}\,dv$

$\ln|x| = \int \frac{1}{v(v+1)}\,dv$

partial fractions

$1 = A(v+1) + B(v)$ $\quad v=0$

$\frac{1}{0+1} = A = 1$ $\quad v=-1$

$1 = B(-1)$

$B = -1$

$\ln|x| = \int \frac{1}{v} - \frac{1}{v+1}\,dv$

$\ln|x| = \ln|v| - \ln|v+1| + c$

$e^{\ln|x|} = e^{\ln\left|\frac{v}{v+1}\right| + c}$

$y = vx$

$v = \frac{y}{x}$

$x = A\left(\frac{v}{v+1}\right)$

$x = A\,\frac{\frac{y}{x}}{\left(\frac{y}{x}+1\right)}$

$x(y+x) = Ay$

$xy + x^2 = Ay$

$xy - Ay = -x^2$

$\boxed{y = \frac{x^2}{A-x}}$ for some constant A.

answer

$\boxed{y = \frac{x^2}{A-x}}$

sorry

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\frac{-2 \pm \sqrt{4-4(1)(-1)}}{2}$

$\frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$

$-1 + \sqrt{2}$

$\frac{\sqrt{4-4(1)(-1)}}{2} = \frac{-2 \pm \sqrt{8}}{2}$

$\frac{-2 \pm 2\sqrt{2}}{2}$

$v = 3$

$-1 \pm \sqrt{2}$

$r_1 = (v + 1 + \sqrt{2})$

$r_2 = (v + 1 - \sqrt{2})$

$1 + \sqrt{2} - 2$

I am dumb and thought

$v^2 - 2v - v =$

$v^2 - 2v - 1$

and then got confused as to why r_1 and r_2 didn't multiply out

don't mind me.

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4. Consider the autonomous equation:

$$y' = y(y-2)e^y$$

✓ (a) (2 points) Find the equilibrium solutions of the above differential equations.

$y' = 0$ at

$y = 0, 2$

so these are the equilibria

✓ (b) (3 points) Determine the stability of the equilibrium solutions.

at $y < 0$ $y' \geq 0$

at $0 < y < 2$, $y' < 0$

at $y > 2$, $y' > 0$

so the equilibrium at $y = 0$ is asymptotically stable.

The equilibrium at $y = 2$ is unstable.

✓ (c) (5 points) Prove that if $y(t)$ is a solution and $y(0) = 1$, then $0 < y(t) < 2$ for all $t \in (-\infty, \infty)$.

we know that

$y = 0$ is a solution because ~~if~~ $y = 0$, then $y' = 0$. so

$y' = 0(0-2)e^0$

✓ $0 = 0$ ✓

and $y = 2$ is a solution ble if $y = 2$, then $y' = 0$.

$y' = 2(2-2)e^y$

✓ $0 = 0$ ✓

$y' = y(y-2)e^y$

setting $f(y,t) = y(y-2)e^y$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y^2e^y - 2ye^y) = 2ye^y + y^2e^y - 2e^y - 2ye^y$$

This is continuous for $t \in (-\infty, \infty)$

$y(0) = 1$ which is $0 < y(0) < 2$, and we know that $\frac{\partial f}{\partial y}$ and y are continuous for all t.

so $y(t)$ must be a unique solution for $t = (-\infty, \infty)$ so it can't cross other solutions. Thus, $0 < y(t) < 2$ for all t.