19F-MATH33B-2 Midterm 1

[redacted]

TOTAL POINTS

40 / 45

QUESTION 1

Separable Equations 10 pts

1.1 SE1 6.5 / 5

✓ + **2 pts** **Separated variables correctly**

✓ + **1 pts** **Integrated with respect to y correctly**

✓ + **1 pts** **Integrated with respect to x correctly**

✓ + **1 pts** **+ C**

+ **0.5 pts** Bonus: mentioned change of variables or used u-sub

✓ + **1.5 pts** **Bonus: mention and explicitly state change of variables**

+ **0 pts** No work

+ **0.5 pts** Small error when integrating with respect to y

+ **1.5 pts** Minor mistake when separating variables

+ **0.5 pts** Small error when integrating with respect to x

1.2 SE2 5.5 / 5

✓ + **2 pts** **Correctly separated variables**

✓ + **1 pts** **Integrated with respect to x correctly**

✓ + **1 pts** **Integrated with respect to t correctly**

✓ + **1 pts** **+C**

✓ + **0.5 pts** **Bonus: mentioned change of variables or used u-sub**

+ **1.5 pts** Bonus: mention change of variables and explicitly state formula

+ **0.5 pts** Minor mistake when integrating with respect to x

+ **1.5 pts** Minor mistake when separating variables

+ **0.5 pts** Minor mistake when integrating with respect to t

QUESTION 2

Exact Differential Equations 12 pts

2.1 EDE1 4 / 4

✓ + **4 pts** **Correct**

+ **2 pts** small error

+ **1 pts** dP/dy=dQ/dx only

+ **1 pts** Said dQ/dy=dP/dx, but then proceeded correctly

+ **0 pts** incorrect

2.2 EDE2 8 / 8

✓ - **0 pts** **Correct**

QUESTION 3

3 First Order Linear Equation 8 / 8

✓ - **0 pts** **Correct: Found general solutions, but not necessarily the solution to the IVP.**

Your Constant C depends on t, and hence is not really constant. Set t=0 to get constant.

QUESTION 4

Existence and Uniqueness 10 pts

4.1 Existence 3 / 5

✓ - **2 pts** **Large error in computation or explanation**

4.2 Uniqueness 5 / 5

✓ - **0 pts** **Correct**

QUESTION 5

5 Bonus Question 0 / 5

+ **5 pts** Correct.

✓ + **0 pts** **Need to start with solutions to implicit equation, then prove using multivariable chain rule that solutions to implicit equation are also solutions**

of the differential equation.

+ **1 pts** Mentioned chain rule, but need additional explanation or used incorrect argument.

+ **0 pts** Not attempted.

+ **2 pts** Correct direction, but need more explanation. What do your computations imply?

Math 33B
Differential Equations

Midterm 1

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, section and UID in the space below.

Name: [redacted]
Student ID number: [redacted]
Section: 2B
Number of extra pages: ____

Question	Points	Score
1	10	
2	12	
3	8	
4	10	
Total:	40	

Bonus $+1.5pts$ each time you state and use the change of variables formula **wherever appropriate.**

Problem 5 (in the last page) is a bonus problem for $5pts$.

[redacted]

Problem 1.

Find the general solution of the following differential equations

(a) [5pts.]

$$\frac{dy(x)}{dx} = e^{x+4y(x)},$$

$$\frac{dy(x)}{dx} = e^{x} e^{4y(x)}$$

$$\frac{1}{e^{4y(x)}} \frac{dy(x)}{dx} = e^{-4y(x)} \left(\frac{dy(x)}{dx} \right) = e^{x}$$

COV Formula ($\int j(u(x)) \frac{du(x)}{dx} dx = J(u(x)) + C$)

$\therefore$ $j(u(x)) = e^{-4y(x)}$ & $u(x) = y(x)$

$$\int e^{-4y(x)} \frac{dy(x)}{dx} dx = \int e^{x} dx$$

$$-\frac{1}{4} e^{-4y(x)} = e^{x} + C$$

$$e^{-4y(x)} = -4e^{x} + C_2 \qquad (C_2 = -4C)$$

$$\boxed{\frac{-1}{4} \ln(-4e^{x} + C_2) = y(x)}$$

(b) [5pts.]

$$\frac{dx(t)}{dt} = \frac{t^4 + \sin(t)}{\cos(x(t)) + 2},$$

$$(\cos(x(t)) + 2) \left(\frac{dx(t)}{dt} \right) = t^4 + \sin(t)$$

COV Formula (see above; $j(u(x)) = \cos(x(t)) + 2$ & $u(x) = x(t)$)

$$\int (\cos(x(t)) + 2) \left(\frac{dx(t)}{dt} \right) dt = \int t^4 + \sin(t) \, dt$$

$$\sin(x(t)) + 2(x(t)) = \frac{t^5}{5} - \cos(t) + C$$

$$\boxed{\sin(x(t)) + 2(x(t)) = \frac{t^5}{5} - \cos(t) + C}$$

[redacted]

Problem 2.

(a) [4pts.] Consider the differential equation

$$x - x\big(y(x)\big)^2 + \big(y(x) - kx^2 y(x)\big)\frac{dy(x)}{dx} = 0$$

Using the definition of exactness of a differential equation, find a value for the unknown constant k so that the above differential equation is exact. You **do not need to solve** the differential equation.

$P = \frac{dF}{dx} = x - xy^2$ (where $y = y(x)$)

$Q = \frac{dF}{dy} = y - kx^2 y$

To be exact:

$\frac{dP}{dy} = \frac{dQ}{dx}$ (or $\text{curl}_z F = 0$)

$\frac{dP}{dy} = -2xy$ & $\frac{dQ}{dx} = -2kxy$

$-2xy = -2kxy$

$\boxed{\therefore k \text{ can be } 1}$

(b) [8pts.] Check if the following differential equation is exact. If so, find the general solution of the differential equation.

$$12xy(x) - 6\big(x^2 - \underbrace{(y(x))^2}\big)\frac{dy(x)}{dx} = 0$$

$-6x^2 + 6y^2$

$P = 12xy$ $(y = y(x))$

$Q = -6x^2 + 6y^2$ $(y = y(x))$

$\frac{dP}{dy} = 12x$ $\qquad \frac{dQ}{dx} = -12x$

As $12x \neq -12x$, the DE is not exact (as F is not a potential function) $- \text{curl}_z F \neq 0$ & $\frac{dP}{dy} \neq \frac{dQ}{dx}$

$\boxed{\text{Correction}}$: $+6\big(x^2 - (y(x))^2\big)\frac{dy(x)}{dx}$

$P = 12xy$ $(y = y(x))$

$Q = 6x^2 - 6y^2$ $(y = y(x))$

$\frac{dP}{dy} = 12x$ $\quad \frac{dQ}{dx} = 12x$ $\quad$ ($\therefore$ as $\frac{dP}{dy} = \frac{dQ}{dx}$, DE is exact)

$\int P(x,y)\,dx = 6x^2 y + \phi(y) = F(x,y)$

$\frac{dF(x,y)}{dy} = Q = 6x^2 + \phi'(y) = 6x^2 - 6y^2$ $\quad$ ($\therefore \phi'(y) = -6y^2$

$\int \phi'(y)\,dy = \int -6y^2\,dy = -2y^3 + C$)

$\therefore \boxed{F(x,y) = 6x^2 y - 2y^3 = C}$

[redacted]

Problem 3. *8pts.*

Let $a(t)$ and $f(t)$ be continuous functions and let y_0 be some real number. Using any method of your choice, solve the following initial value problem,

$$\frac{dy(t)}{dt} = a(t)y(t) + f(t), \quad y(0) = y_0$$

You must show all the steps required to solve the problem.

$$-a(t)y(t) + \frac{dy(t)}{dt} = f(t)$$

Using an integrating factor: $u = e^{-\int a(t)dt}$

$$\Rightarrow \underbrace{\left(-a(t)y(t) + \frac{dy(t)}{dt}\right)e^{-\int a(t)dt}}_{\text{Recognize this is a product rule expanded out}} = f(t)$$

$$\Rightarrow \frac{d}{dt}\left(e^{-\int a(t)dt}\, y(t)\right) = e^{-\int a(t)dt} f(t) \Rightarrow \text{integrating both sides}$$

$$\int_{t}^{t_0} \frac{d}{dt}\left(e^{-\int a(t)dt}\, y(t)\right) dt = \int_{t}^{t_0} e^{-\int a(t)dt} f(t)\, dt$$

$$e^{-\int a(t)dt}\, y(t) = \int_{t}^{t_0} e^{-\int a(t)dt} f(t)\, dt + C$$

General solution

$$\boxed{y(t) = e^{\int a(t)dt} \int_{t}^{t_0} e^{-\int a(t)dt} f(t)\, dt + Ce^{\int a(t)dt}}$$

$$\Rightarrow y(t) = \int_{t}^{t_0=0} f(t)\, dt + Ce^{\int_t^0 a(t)dt} = y_0 \text{ (at } t_0 = 0)$$

Constant Value

$$\boxed{C = \frac{y_0 - \int_t^{t_0=0} f(t)\, dt}{e^{\int_t^{t_0=0} a(t)dt}}}$$

[redacted]

Problem 4.

Consider the initial value problem (IVP) : $\frac{dp(t)}{dt} = (p(t)+4t)^{\frac{1}{3}}$, $p(1) = 4$.

(a) [5pts.] Can we guarantee that this IVP has a solution? If not, can we use the existence theorem to say that no solution to this IVP exists. Justify your answer.

$f(t) = \sqrt[3]{p(t)+4(t)}$

The function is continuous so long as the radical's inside is positive, ~~therefore, as long as~~ ~~$p(t) \geq -4t$~~, the function is continuous. As there is an interval where the RHS is continuous, and as the initial condition $(t=1, p(t)=4)$ satisfies and is within this continuous interval $(\sqrt[3]{4+4} = \sqrt[3]{8} = 2)$, we can say that this IVP does indeed have a solution for some δ such that the solution lies within the interval $[t_0-\delta, t_0+\delta]$, thus a solution exists.

(b) [5pts.] Can we guarantee that a unique solution of the IVP exists? If not, can we say that there exist multiple solutions to this IVP by applying the uniqueness theorem? Justify your answer.

$\frac{df}{dp} = \frac{1}{3}(p(t)+4t)^{-\frac{2}{3}}\left(\frac{dp(t)}{dt}\right)$

$\hookrightarrow \frac{1}{3(p(t)+4t)^{\frac{2}{3}}}$

For uniqueness theorem, we must look at both $f(t,p(t)) = \sqrt[3]{p(t)+4t}$ and also its partial derivative $\frac{df}{dp} = \frac{1}{3(p(t)+4t)^{2/3}}$; While we know that f is continuous so long as the inside of the radical is greater than zero, for f_p (its partial derivative with respect to p), it is continuous as long as the radical is positive (so unlike f, it cannot be 0 within the radical). As the initial condition fulfills the sufficient conditions $(t_0=1, p(t_0)=4)$ and such that the derivative and f are continuous, we can thus construct a rectangle $[b,a]\times[d,c]$ that contains the initial conditions where within that rectangle, we can guarantee a unique solution (as sufficient conditions are satisfied).

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Problem 5(Bonus problem) 5*pts.*
Show that if the differential equation,

$$P(y,t(y)) + Q(y,t(y))\frac{dt(y)}{dy} = 0$$

is exact, then there exists a function $R(p,q)$ such that the general solution $t(y)$ of the differential equation is given by

$$R(y,t(y)) = C$$

If exact:

$$\frac{dP}{dy} = \frac{dQ}{dt} \quad (\text{where } t = t(y))$$

$$\Rightarrow \frac{dP}{dy} - \frac{dQ}{dt} = 0 = \text{curl}_z F \text{, where } F \text{ is some potential function.}$$

Multiplying through by dy (using differential form notation):

$$P(y,t(y)) + Q(y,t(y))\frac{dt(y)}{dy} = 0 \quad \text{becomes}$$

$$P(y,t(y))\,dy + Q(y,t(y))\,dt(y) = 0$$

$$= \langle P(y,t(y)), Q(y,t(y))\rangle \cdot \langle dy, dt(y)\rangle = 0$$

$$= \nabla F(y,t(y)) \cdot \langle dy, dt(y)\rangle = 0$$

N.B. This expression above has a solution at all level curves, meaning at level height ($\therefore$ there must be a case where there is a potential function $F = R(y,t(y))$ that satisfies this condition)