1. (10 pts) Solve the initial value problem

$$y^2 - (x^2 + 2xy)y' = 0, \qquad y(1) = 1$$

(Hint: Look for an integrating factor that depends only on x.)

$$\underbrace{y^2}_{P} dx + \underbrace{-(x^2+2xy)}_{Q} dy = 0$$

$$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = -2x-2y$$

$$h = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{2y-(-2x-2y)}{-(x^2+2xy)} = \frac{2(x+2y)}{-x(x+2y)} = \frac{2}{-x} \quad \checkmark$$

Since h is only a function of x, we let

$$u(x) = e^{\int h(x)\,dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln(x^{-2})} = x^{-2} = \frac{1}{x^2}$$

Multiply the original equation by $\frac{1}{x^2}$:

$$\frac{y^2}{x^2}dx + -\left(1+2\frac{y}{x}\right)dy = 0$$

Want $f(x,y)$ with $\frac{\partial f}{\partial x} = \frac{y^2}{x^2}$ and $\frac{\partial f}{\partial y} = -1-2\frac{y}{x}$

$$f(x,y) = \int \frac{y^2}{x^2}dx = y^2\int x^{-2}dx = -y^2\cdot x^{-1} + C(y) = -\frac{y^2}{x} + C(y)$$

$$\frac{\partial f}{\partial y} = \cancel{\frac{-2y}{x}} + C'(y) = -1 \cancel{-2\frac{y}{x}} \Longrightarrow C'(y) = -1$$

$$C(y) = \int -1\,dy = -y + C$$

So we can let $f(x,y) = -\frac{y^2}{x} - y$.

General Solution: $-\frac{y^2}{x} - y = C$

Initial condition $y(1)=1$: $-\frac{1^2}{1} - 1 = C \Rightarrow C = -2$

$$\boxed{-\frac{y^2}{x} - y = -2}$$

or $y^2 + xy - 2x = 0$

$$y = \frac{-x \pm \sqrt{x^2+8x}}{2} \qquad y(1) = \frac{-1\pm 3}{2},$$

so $+$, not $-$.

$$\boxed{y = \frac{1}{2}\left(\sqrt{x^2+8x} - x\right)}$$

2. (10 pts) Consider the differential equation

$$y' = e^{-y}(2y^4 + 1y^3 - 6y^2).$$

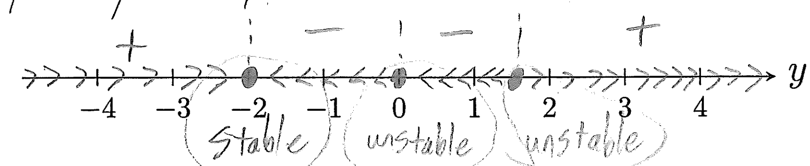
(a) (5 pts) Find the equilibria, draw the phase line, and classify each equilibrium as stable or unstable.

$e^{-y}(2y^4 + y^3 - 6y^2) = 0$

$e^{-y} \cdot y^2(2y^2 + y - 6) = 0$

$e^{-y} y^2 (2y-3)(y+2) = 0 \quad \longrightarrow \quad y=0,\ y=-2,\ y=\frac{3}{2}$

Phase line:



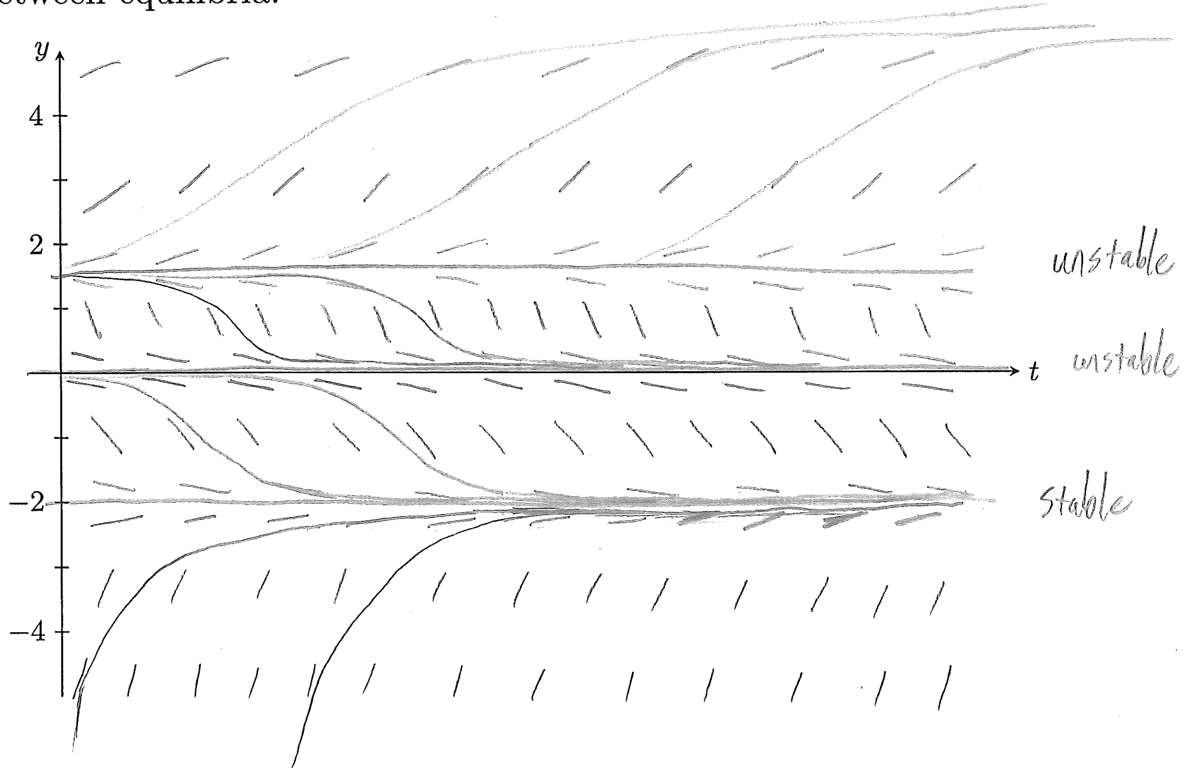
Plugging in: $y=-3$: $(+)(+)(-)(-) = +$

$y=-1$: $(+)(+)(-)(+) = -$

$y=1$: $(+)(+)(-)(+) = -$

$y=3$: $(+)(+)(+)(+) = +$

(b) (5 pts) Draw a rough sketch of the direction field, including the equilibrium solutions, and sketch a few solution curves in each region between equilibria.



3. (10 pts) Recall that Newton's Law of Cooling says that the temperature T of an object will change according to the following differential equation:

$$T' = -k(T - A)$$

where A is the temperature of the object's surroundings, and k is a positive constant that depends on the object. You take a pie out of the oven at 200° F. From previous pies you've baked, you have calculated that $k = 0.1/\text{min}$. The temperature in the kitchen is 70° F, but is dropping at 1°/min. Find an equation for the temperature T of the pie as a function of time.

$A(0) = 70,\ A' = -1$

So $A(t) = 70 - t$

$$T' = -0.1(T - (70 - t))$$
$$= -0.1T + 7 - 0.1t$$

$T' + 0.1T = 7 - 0.1t$ Linear. Integrating factor: $u = e^{\int 0.1\,dt} = e^{0.1t}$

$$e^{0.1t}T' + 0.1e^{0.1t}T = (7 - 0.1t)e^{0.1t}$$
$$\Downarrow$$
$$e^{0.1t}T = \int (7 - 0.1t)e^{0.1t}\,dt = \int 7e^{0.1t}\,dt - \int 0.1te^{0.1t}\,dt$$
$$= 7\cdot\frac{1}{0.1}e^{0.1t} - \left(te^{0.1t} - \int e^{0.1t}\,dt\right)$$

$\leftarrow$ IBP! $u = t$ $dv = 0.1e^{0.1t}dt$, $du = dt$ $v = e^{0.1t}$

$$= 70e^{0.1t} - te^{0.1t} + \frac{1}{0.1}e^{0.1t} + C$$

$T(0) = 200 \Rightarrow 1\cdot 200 = 70 - 0 + 10 + C \Rightarrow C = 120$

$$\cancel{e^{0.1t}}\cdot T = 70\cancel{e^{0.1t}} - t\cancel{e^{0.1t}} + 10\cancel{e^{0.1t}} + 120$$

$$\boxed{T = 80 - t + 120e^{-0.1t}}$$

4. (10 pts) A 100 L tank initially contains 10 L of a salt-water solution at a concentration of 5 g of salt per liter of water. At time $t = 0$, water begins flowing into the tank through two pipes: pipe A contains pure water and flows at 1 L/min, and pipe B contains another salt-water solution at a concentration of 3 g/L and flows at 2 L/min. At the same time, a drain is opened at the bottom of the tank that lets the (perfectly mixed) salt-water solution flow out at 2 L/min.

(a) (5 pts) Set up a differential equation describing the *amount* of salt y in the tank at time t.

Volume of solution in tank: $V(0)=10$, $V'=(1+2)-2=1 \Rightarrow V(t)=10+t$

Rate in: $(3\ \text{g/L})\cdot(2\ \text{L/min}) = 6\ \text{g/min}$

Rate out: $(2\ \text{L/min})\cdot\left(\frac{y}{10+t}\right) = \frac{2y}{10+t}\ \text{(g)/min}$

$$y' = 6 - \frac{2y}{10+t} \qquad y(0) = (10\ \text{L})\cdot(5\ \text{g/L}) = 50\ \text{g}$$

(b) (5 pts) Solve the differential equation from part (a). How much salt will be in the tank at the instant that it is full?

$$y' + \frac{2}{10+t}y = 6$$

Linear. Integrating factor:

$$u = e^{\int \frac{2}{10+t}dt} = e^{2\ln(10+t)} = (10+t)^2$$

$$(10+t)^2 y' + 2(10+t)y = 6(10+t)^2$$

$$\Downarrow$$

$$(10+t)^2 y = \int 6(10+t)^2 dt = 6\cdot\tfrac{1}{3}(10+t)^3 + C = 2(10+t)^3 + C$$

$y(0)=50$: $\quad 10^2\cdot 50 = 2\cdot(10)^3 + C \Longrightarrow 5000 = 2000 + C \Rightarrow C = 3000$

$$(10+t)^2 y = 2(10+t)^3 + 3000$$

$$y = 2(10+t) + 3000(10+t)^{-2} = \boxed{20 + 2t + \frac{3000}{(10+t)^2}}$$

Tank full when $V = 10+t = 100 \Longrightarrow t = 90$

$$y(90) = 20 + 2\cdot 90 + \frac{3000}{(100)^2} = 20 + 180 + \frac{3000}{10000} = \boxed{200.3\ \text{g}}$$