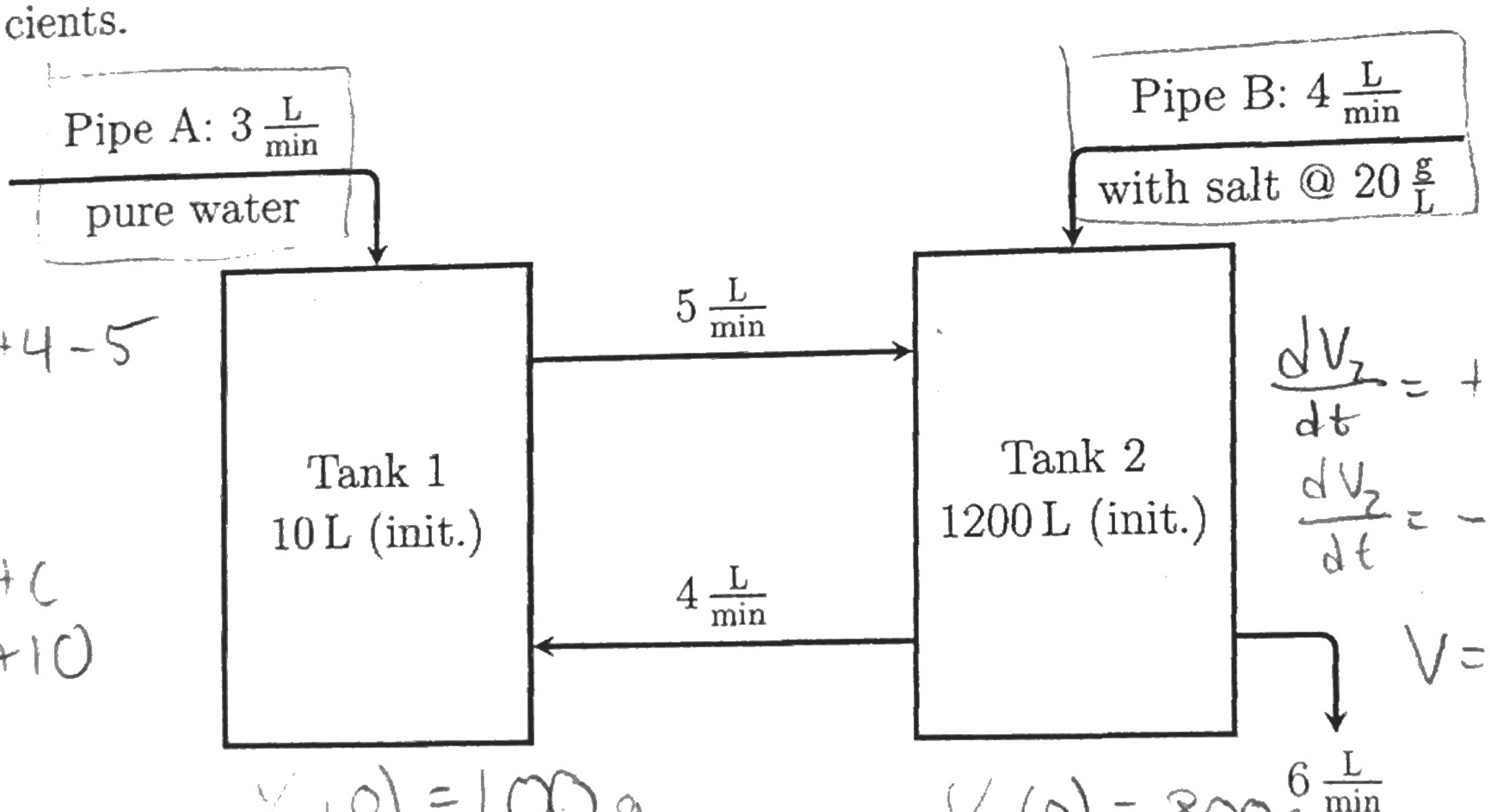
1. (10 pts) Tank 1 initially contains 10 L of water with 100 g of salt. Tank 2 initially contains 1200 L of water with 800 g of salt. The tanks are connected by pipes with flow rates as shown in the diagram below. Note that the volume of solution in each tank is changing! Note also that the water entering tank 1 through pipe A is pure water, whereas the solution entering tank 2 through pipe B contains salt at a concentration of 20 $\frac{\text{g}}{\text{L}}$. (In all other pipes, the concentration of salt is of course equal to the concentration in the tank that the pipe is coming from.)

Letting y_1 and y_2 be the amounts of salt (in grams) in tank 1 and tank 2 respectively, and using these (and t) as your *only* variables, **write down a system of differential equations, with initial conditions,** to model this situation.

Is the resulting system linear? If so, write it in matrix form, and also specify if it is homogeneous, and whether or not it has constant coefficients.



$= +3+4-5$

$\frac{dV_1}{dt} = \frac{2L}{min}$

$V = 2t + C$

$V = 2t + 10$

$\frac{dV_2}{dt} = +4+5-4-6$

$\frac{dV_2}{dt} = -1$

$V = -t + 1200$

$y_1(0) = 100\,g$

$y_2(0) = 800\,g$

(Answer on the following page.)

$$\frac{dy_1}{dt} = \frac{4L}{min}\left(\frac{y_2}{-t+1200}\right) - \frac{5L}{min}\frac{y_1}{(2t+10)}$$

$$\frac{dy_2}{dt} = \left(\frac{5y_1}{2t+10} + \frac{4L}{min}\left(\frac{20g}{L}\right)\right) - \frac{10L}{min}\frac{y_2}{(-t+1200)}$$

(Problem 1 continued...)

$$\frac{dy_1}{dt} = -\frac{5y_1}{(2t+10)} + \frac{4y_2}{1200-t}; \text{ where } y_1(0) = 100$$

$$\frac{dy_2}{dt} = \frac{5y_1}{2t+10} - \frac{10y_2}{1200-t} + (4)(20); \quad y_2(0) = 800$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -\frac{5}{2t+10} & \frac{4}{1200-t} \\ \frac{5}{2t+10} & -\frac{10}{1200-t} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 80 \end{pmatrix}$$

Linear?	YES	NO
Homogeneous?	YES	NO
Constant coefficients?	YES	NO

2. (10 pts) Find the general solution of the system

$$\mathbf{y}' = \begin{bmatrix} -7 & -4 \\ 4 & 1 \end{bmatrix} \mathbf{y}.$$

$$\det\begin{pmatrix} -7-\lambda & -4 \\ 4 & 1-\lambda \end{pmatrix} = (-7-\lambda)(1-\lambda) + 16$$

$$= -7 + 6\lambda + \lambda^2 + 16$$

$$= \lambda^2 + 6\lambda + 9$$

$$= (\lambda+3)(\lambda+3)$$

$$\lambda = -3$$

$$\boxed{\vec{y} = c_1 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} +} \; c_2 e^{-3t}\left[\begin{pmatrix} -\frac{5}{4} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} t\right]$$

$$\begin{pmatrix} -7-(-3) & -4 \\ 4 & 1-(-3) \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}$$

$$\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-4V_1 - 4V_2 = 0$$

$$-4V_1 = 4V_2$$

$$V_1 = -V_2$$

One possible eigen vector: $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{v}$

$$\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$-4w_1 - 4w_2 = 1$$

$$4w_1 + 4w_2 = -1$$

$$w_1 + w_2 = -\frac{1}{4}$$

$$w_1 = -\frac{1}{4} - w_2$$

$$\vec{w} = \begin{pmatrix} -\frac{5}{4} \\ 1 \end{pmatrix}$$

$$\vec{y} = \begin{bmatrix} e^{4t}\cos t - e^{4t}\sin t + i(e^{4t}\cos t + e^{4t}\sin t) \\ -\cos t + i(\cancel{-\sin t}) \end{bmatrix}$$

General soln: $\vec{y} = C_1 \begin{bmatrix} e^{4t}\cos t - e^{4t}\sin t \\ -\cos t \end{bmatrix} + C_2 \begin{bmatrix} e^{4t}\cos t + e^{4t}\sin t \\ -\sin t \end{bmatrix}$

3. (10 pts) Find the general solution of the system in $\mathbb{R}$ #'s

$$\mathbf{y}' = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} \mathbf{y}.$$

$$\det\begin{pmatrix} 5-\lambda & 2 \\ -1 & 3-\lambda \end{pmatrix} = (5-\lambda)(3-\lambda) + 2$$

$$= 15 - 8\lambda + \lambda^2 + 2$$

$17 \times 4 = 68$

$$= \lambda^2 - 8\lambda + 17$$

$$\lambda = \frac{8 \pm \sqrt{64 - 4(17)}}{2} = \frac{8 \pm \sqrt{-4}}{2} = 4 \pm i$$

$\lambda = 4+i$

$$\begin{bmatrix} 5-(4+i) & 2 \\ -1 & 3-(4+i) \end{bmatrix} = \begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$\begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(1-i)V_1 + 2V_2 = 0$

One soln:

$$\vec{y} = e^{(4+i)t}\begin{bmatrix} 1+i \\ -1 \end{bmatrix}$$

$$\vec{y} = e^{4t+it}\begin{bmatrix} 1+i \\ -1 \end{bmatrix}$$

$$\vec{y} = e^{4t}e^{it}\begin{bmatrix} 1+i \\ -1 \end{bmatrix}$$

or $\vec{V} = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$

$$\vec{y} = e^{4t}(\cos t + i\sin t)\begin{bmatrix} 1+i \\ -1 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} (e^{4t}\cos t + i e^{4t}\sin t)(1+i) \\ -\cos t - i\sin t \end{bmatrix} = \begin{bmatrix} e^{4t}\cos t + i e^{4t}\cos t + i e^{4t}\sin t - e^{4t}\sin t \\ -\cos t - i\sin t \end{bmatrix}$$

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4. (10 pts) Match each of the following systems of differential equations with the phase portraits on the next page, and classify the type of equilibrium point at the origin for each one.
(1 pt each)

Linear System	Plot	Type of Equilibrium
1. $\mathbf{y}' = \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix} \mathbf{y}$	E	Stable ✓
2. $\mathbf{y}' = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \mathbf{y}$	D	Saddle point ✓
3. $\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix} \mathbf{y}$	✓ C	unstable, ~~node~~ spiral source ✗
4. $\mathbf{y}' = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \mathbf{y}$	A	unstable ✓
5. $\mathbf{y}' = \begin{bmatrix} -1 & 8 \\ -2 & -1 \end{bmatrix} \mathbf{y}$	B	Stable, ~~node~~ spiral sink ✗

$\lambda = \frac{-5 \pm \sqrt{(5 \cdot 4 \cdot)}}{2}$ ± 1

1. $\det\begin{pmatrix} -4-\lambda & 1 \\ -2 & -1-\lambda \end{pmatrix} = (-4-\lambda)(-1-\lambda) + 2 = 6 + 5\lambda + \lambda^2$
$(\lambda+2)(\lambda+3)$

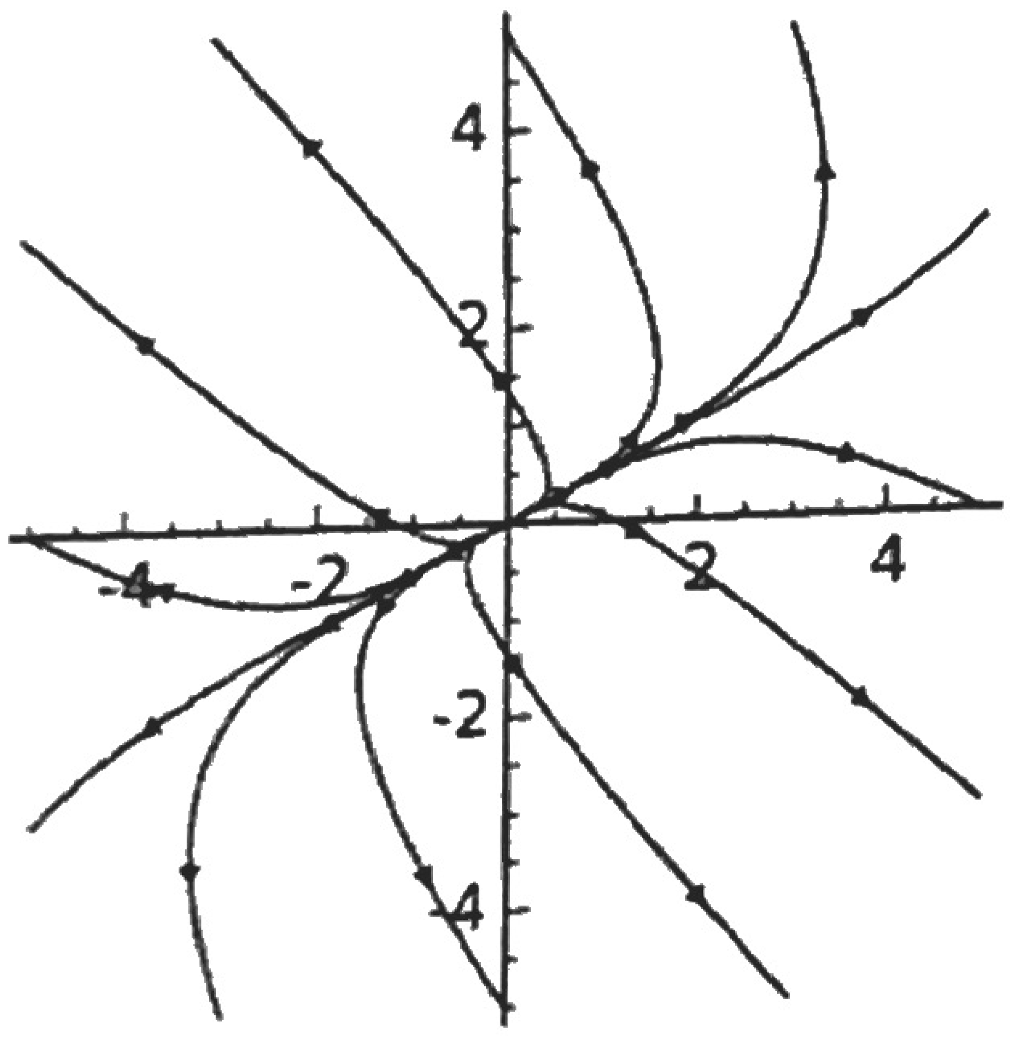
2. $\det\begin{pmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{pmatrix} = (3-\lambda)(1-\lambda) - 8 = -5 - 4\lambda + \lambda^2$ $\lambda = 5$
$(\lambda-5)(\lambda+1)$ $\lambda = -1$

3. $\det\begin{pmatrix} 3-\lambda & 2 \\ -4 & -1-\lambda \end{pmatrix} = (3-\lambda)(-1-\lambda) + 8 = 5 - 2\lambda + \lambda^2$
$2 \pm \sqrt{4 - 4(5)}$

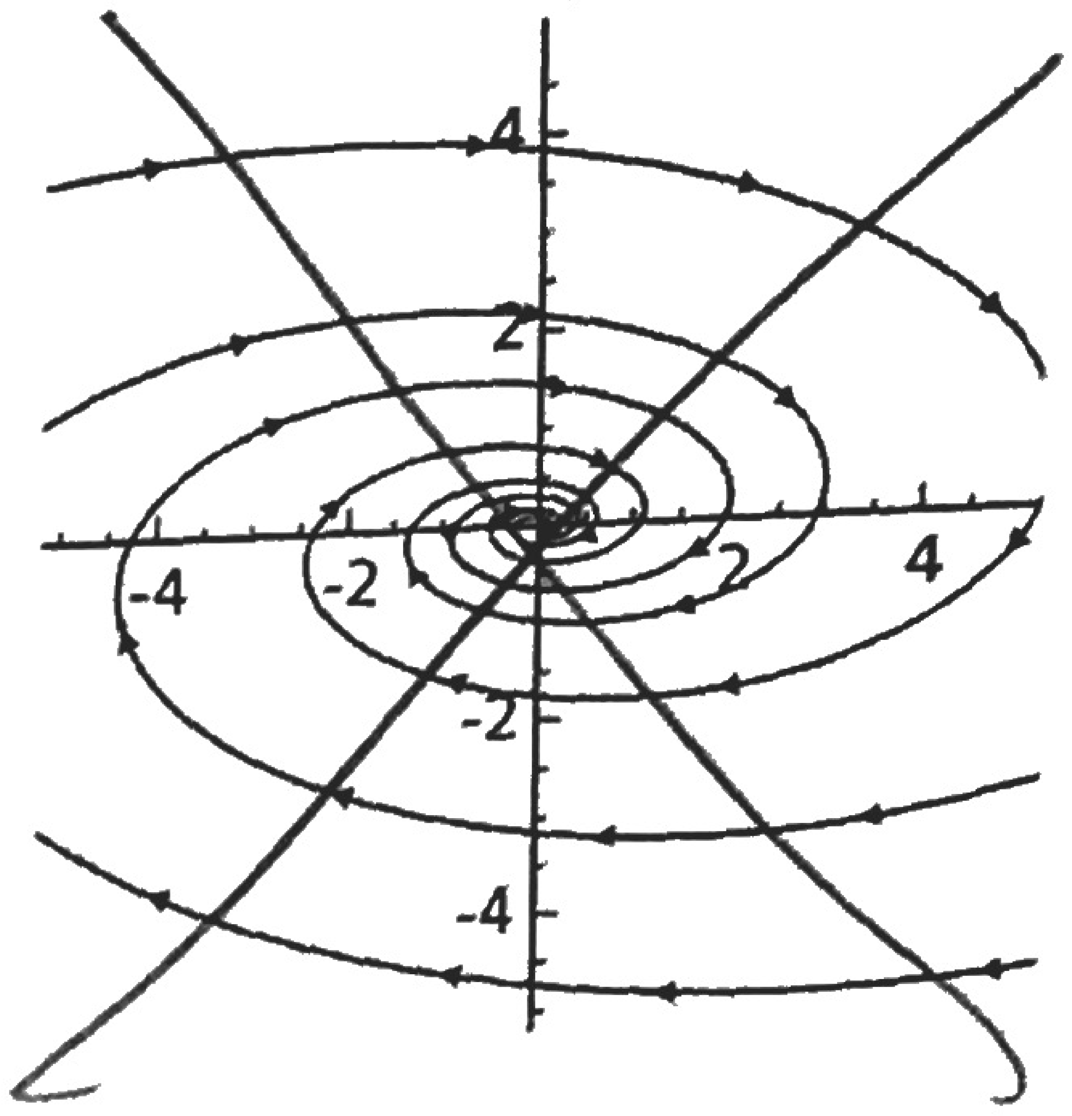
4 $\det\begin{pmatrix} 2-\lambda & -2 \\ -1 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) - 2 = 4 - 5\lambda + \lambda^2$
$(\lambda-4)(\lambda-1)$

5 $\det\begin{pmatrix} -1-\lambda & 8 \\ -2 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1-\lambda) + 16 = 17 + 2\lambda + \lambda^2$

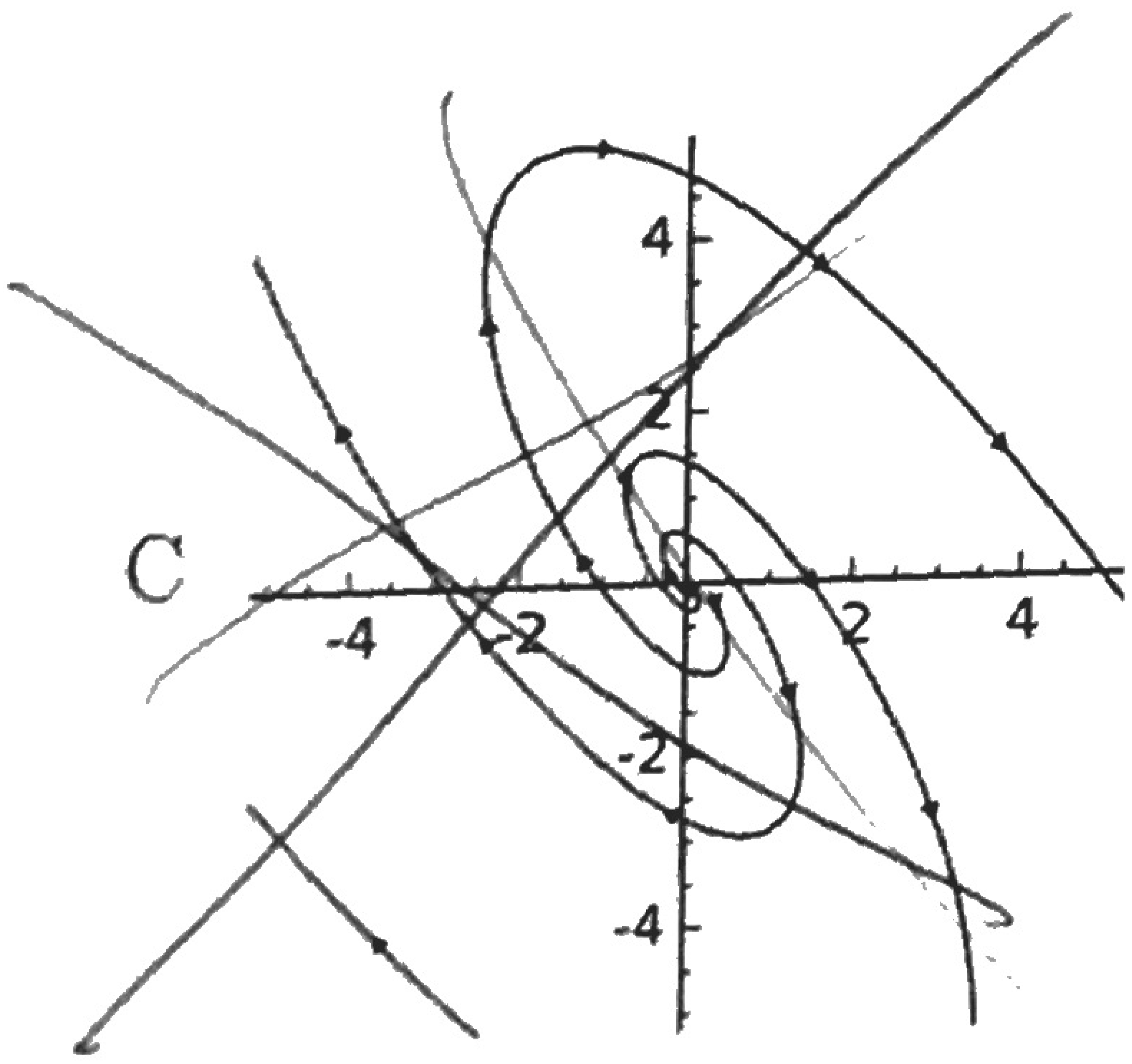
A



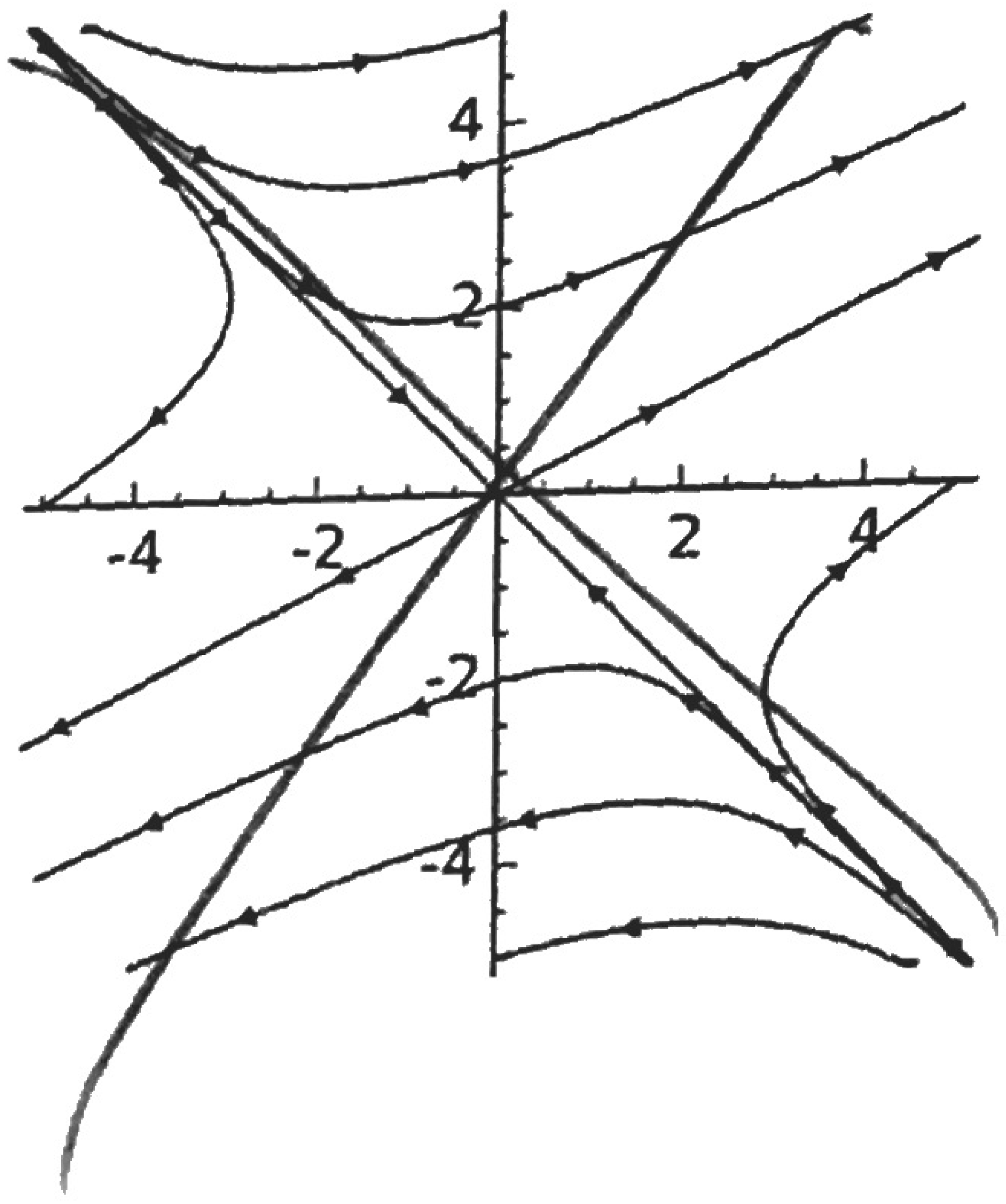
B



C



D



E