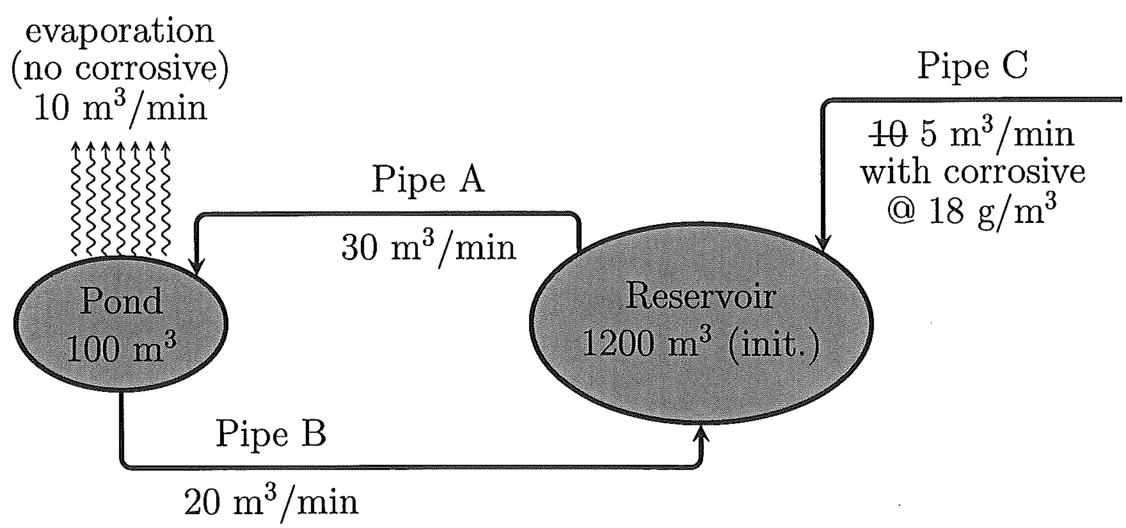
1. (10 pts) At a nuclear power plant, spent fuel rods are kept in a small cooling pond containing 100 m^3 of pure water. Due to the heat produced by these fuel rods, the water in this pond evaporates at a rate of 10 m^3/min. Since it is essential that this pond be kept full and relatively cool, this water is exchanged with water from a larger reservoir outside the plant: water is pumped from that reservoir to the pond through pipe A at 30 m^3/min, and pumped from the bottom of the pond back to the reservoir through pipe B at 20 m^3/min. The reservoir initially contains 1200 m^3 of pure water. To keep the water level in this reservoir constant, water is supposed to be pumped into it through pipe C at a rate of 10 m^3/min. However, an earthquake (at time $t = 0$) causes pipe C's pump to malfunction, so that it only operates at 5 m^3/min. And to make matters worse, the earthquake also causes a crack in this pipe, which introduces a corrosive at a concentration of 18 g/m^3. Assume that this corrosive will not evaporate with the water in the cooling pond.



Set up a system of differential equations *(with initial conditions)* to model the amount of the corrosive in the reservoir and in the pond. Note that you should have 2 or 3 variables. Be sure to state clearly what each of your variables represents.

Rate of change of volume of pond: $+30-10-20=0$, so the pond will have constant volume.

Rate of change of volume of reservoir: $+5+20-30=-5\ \mathrm{m^3/min}$

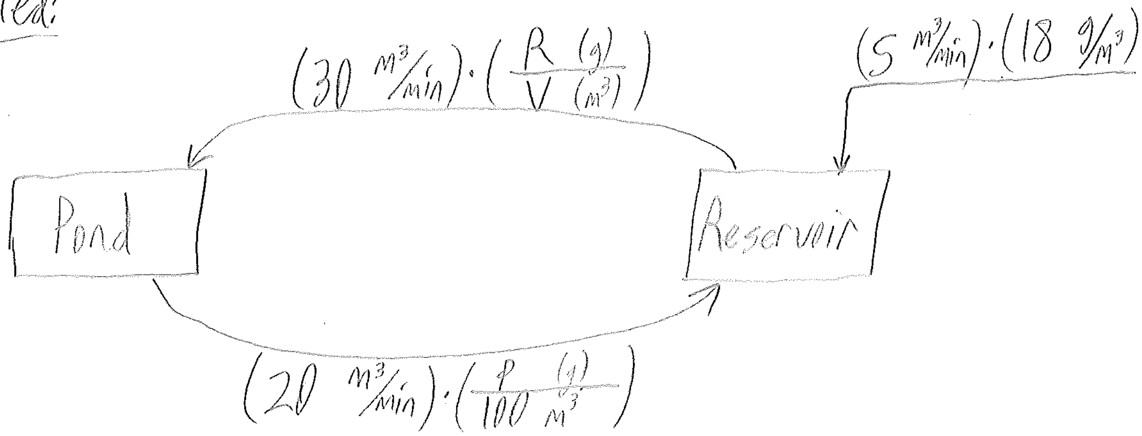
So the volume of the reservoir is changing!

Variables: $V(t)$ = volume of solution in the reservoir (m^3)

$R(t)$ = mass of corrosive in the reservoir (g)

$P(t)$ = mass of corrosive in the pond (g)

Diagram, labeled:



$$\boxed{\begin{cases} V' = -5 & V(0)=1200 \\ R' = +5\cdot 18 - 30\frac{R}{V} + 20\cdot\frac{P}{100} & R(0)=0 \\ P' = +30\frac{R}{V} - 20\frac{P}{100} & P(0)=0 \end{cases}}$$

or, using the fact that $V(t)=1200-5t$:

$$\begin{cases} R' = 5\cdot 18 - \dfrac{30R}{1200-5t} + \dfrac{20P}{100} & R(0)=0 \\ P' = \dfrac{30R}{1200-5t} - \dfrac{20P}{100} & P(0)=0 \end{cases}$$

One method... separable

2. (10 pts) Solve the initial value problem

$$(y^2+4)\sin(x) + 2y\cos(x)y' = 0 \qquad y(\tfrac{\pi}{3}) = 1.$$

This is separable.

$2y\cos x\, y' = -(y^2+4)\sin x$

$\int \frac{2y}{y^2+4}\, dy = \int \frac{-\sin x}{\cos x}\, dx$

$\ln(y^2+4) = \ln(\cos x) + C$

$y^2+4 = e^{\ln(\cos x)+C} = e^{\ln(\cos x)} \cdot e^C = C_1 \cdot \cos x$

where $C_1 = e^C$

When $x = \frac{\pi}{3}$, $y = 1$: $1^2+4 = C_1 \cdot \cos(\frac{\pi}{3})$

$5 = C_1 \cdot \frac{1}{2}$

$C_1 = 10$

$y^2+4 = 10\cos x$

$y^2 = 10\cos x - 4$

$y = \pm\sqrt{10\cos x - 4}$

Since we want $y(\frac{\pi}{3}) = 1$, take the positive part of this.

$$\boxed{y = \sqrt{10\cos x - 4}}$$

Second method: use an integrating factor that is only a function of x to make it exact.

2. (10 pts) Solve the initial value problem

$$\underbrace{(y^2+4)\sin(x)}_{P(x,y)} + \underbrace{2y\cos(x)}_{Q(x,y)}y' = 0 \qquad y(\tfrac{\pi}{3}) = 1.$$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 2y\sin x - -2y\sin x = 4y\sin x \neq 0,$$ so not exact.

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{4y\sin x}{2y\cos x} = 2\tan x$$ is a function of only x

so $$u = e^{\int 2\tan x\,dx} = e^{-2\ln\cos x} = (\cos x)^{-2} = \frac{1}{\cos^2 x}$$ works!

$$\frac{(y^2+4)\sin x}{\cos^2 x} + \frac{2y\cos x}{\cos^2 x}y' = 0$$

$$(y^2+4)\sec x\cdot\tan x + 2y\sec x\cdot y' = 0$$ ← This is exact!

$$f(x,y) = \int (y^2+4)\sec x\tan x\,dx = (y^2+4)\sec x + C(y)$$

$$\frac{\partial f}{\partial y} = 2y\cdot\sec x + C'(y) = 2y\sec x \implies C'(y) = 0 \implies C(y) = 0$$

General solution: $(y^2+4)\cdot\sec x = C$.

$$y^2+4 = C\cdot\cos x$$

When $x = \frac{\pi}{3}$, $y = 1$: $1^2+4 = C\cdot\cos(\frac{\pi}{3}) \implies 5 = C\cdot\frac{1}{2}$

$C = 10$

$$y^2+4 = 10\cos x$$

$$y = \pm\sqrt{10\cos x - 4}$$

Since initial condition has $y = +1$, take positive part: $\boxed{y = \sqrt{10\cos x - 4}}$

Third method: Use an integrating factor that is only a function of x.

$P(x,y) = (y^2+4)\sin x \qquad Q(x,y) = 2y\cos x$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 2y\sin x - {}^{-}2y\sin x = 4y\sin x \quad \neq 0, \text{ so not exact.}$$

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{P} = \frac{4y\,\cancel{\sin x}}{(y^2+4)\,\cancel{\sin x}} = \frac{4y}{y^2+4} \quad \text{is a function of only } y.$$

$$\text{so} \quad u = e^{-\int \frac{4y}{y^2+4}\,dy} = e^{-2\int \frac{2y}{y^2+4}\,dy} = e^{-2\ln(y^2+4)}$$

$$= (y^2+4)^{-2} = \frac{1}{(y^2+4)^2} \quad \text{should work...}$$

$$\frac{(y^2+4)\sin x}{(y^2+4)^2} + \frac{2y\cos x}{(y^2+4)^2}\cdot y' = 0$$

$$\frac{\sin x}{y^2+4} + \frac{2y\cos x}{(y^2+4)^2}\cdot y' = 0 \quad \longleftarrow \text{This is exact!}$$

$$f(x,y) = \int \frac{\sin x}{y^2+4}\,dx = \frac{-\cos x}{y^2+4} + C(y)$$

$$\frac{\partial f}{\partial y} = (-\cos x)\cdot(-1)(y^2+4)^{-2}\cdot 2y + C'(y) = \frac{2y\cos x}{(y^2+4)^2} + C'(y) = \frac{2y\cos x}{(y^2+4)^2}$$

$C'(y) = 0$, so $C(y) = 0$ (or a constant)

General solution: $\dfrac{-\cos x}{y^2+4} = C$

When $x = \frac{\pi}{3}$, $y = 1$:

$$\frac{-\cos(\frac{\pi}{3})}{1^2+4} = C = \frac{-\frac{1}{2}}{5} = -\frac{1}{10}$$

$$\frac{-\cos x}{y^2+4} = -\frac{1}{10} \longrightarrow y^2 + 4 = 10\cos x$$

$$y = \pm\sqrt{10\cos x - 4}$$

Since the initial condition has $y = +1$, take the positive part: $\boxed{y = \sqrt{10\cos x - 4}}$

3. (10 pts) At a barbecue, you leave a cold bottle of beer outside, initially at 40° F. From Newton's Law of Cooling, you know that the rate of change of the bottle's temperature will be proportional to the difference between the its current temperature (T) and the temperature of the air around it (A), and from past experience, you have calculated that the proportionality constant for this brand of beverage is 70% per hour (i.e., 0.7). The air outside is 72° F initially, but it is warming up at 2° F per hour. Set up and solve this differential equation, to find a function for the temperature of the bottle at any time t.

$A'(t) = +2$ and $A(0) = 72$, so

$A(t) = \int 2\,dt = 2t + C$ ... $C = 72$, so

$A(t) = 72 + 2t$

Newton's Law of Cooling: $T'(t) = -k(T - A)$ with $k = 0.7$ and $A = 72 + 2t$

$$T' = -0.7(T - (72 + 2t))$$

$$T' = -0.7T + 0.7(72 + 2t)$$

This is not separable, but it is linear.

$$T' + 0.7T = 0.7\cdot(72 + 2t)$$

$$u = e^{\int 0.7\,dt} = e^{0.7t};$$

$$e^{0.7t}\cdot T' + 0.7e^{0.7t}\cdot T = 0.7\cdot(72 + 2t)e^{0.7t}$$

Integrate both sides: $e^{0.7t}\cdot T = \int 0.7\cdot(72 + 2t)e^{0.7t}\,dt$

$u = 72 + 2t$ $dv = 0.7e^{0.7t}$
$du = 2\,dt$ $v = e^{0.7t}$

$$= (72 + 2t)e^{0.7t} - \int 2e^{0.7t}\,dt$$

$$= (72 + 2t)e^{0.7t} - \frac{2}{0.7}e^{0.7t} + C$$

$$T = 72 - \frac{2}{0.7} + 2t + Ce^{-0.7t}$$

$T(0) = 40$

$$40 = 72 - \frac{2}{0.7} + C \Longrightarrow C = \frac{2}{0.7} - 32$$

$$\boxed{T = 72 - \frac{2}{0.7} + 2t + \left(\frac{2}{0.7} - 32\right)e^{-0.7t}}$$

4. (8 pts) A scientist has introduced a genetic mutation into a population of mice that he is studying in his lab. He asks you to model the spread of the mutation through the population, as they breed normally over the next several generations. You come up with the following model, where $p(t)$ is the *fraction* of mice that carry the genetic mutation:

$$p' = p(p-1)(1-10p)$$

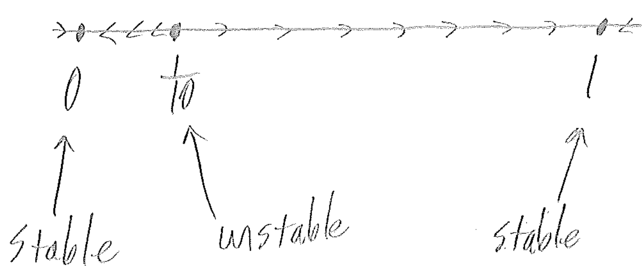
If the scientist originally introduced the mutation into 13% of the mice in his lab, then in the long run (as $t \to \infty$) what fraction of the mice will end up with the mutation? *(Hint: You do not need to solve this differential equation!)*

$p(t)$ should always be between 0 and 1...

The differential equation is autonomous, so look for its equilibrium points:

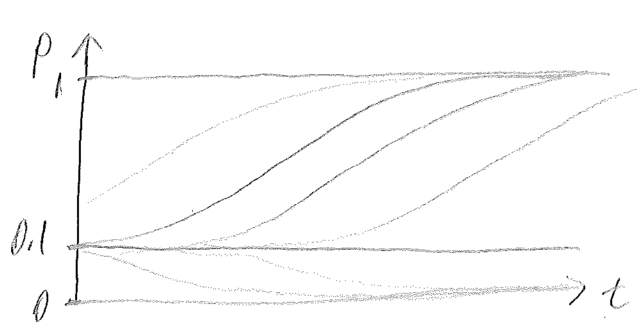
$$p(p-1)(1-10p) = 0 \implies p = 0 \text{ or } p = 1 \text{ or } p = \frac{1}{10} = 0.1$$

Phase portrait:



p	p' (+/−)
−1	+
0.01	−
0.5	+
2	−

$p(t)$ versus t:



So, since $p(0) = 0.13 > 0.1$, p will approach 1 in the long run. That is, eventually $\approx 100\%$ of the mice will carry the mutation.

5. (7 pts) Consider the differential equation

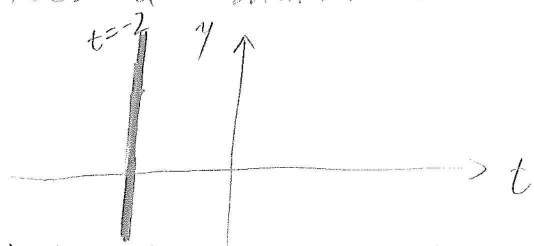
$$(t+2)y' = y^{\frac{2}{3}}. \implies y' = \frac{y^{2/3}}{t+2}$$

(a) For what points (t_0, y_0) does the Existence Theorem guarantee that a solution exists satisfying $y(t_0) = y_0$?

$f(t,y) = \frac{y^{2/3}}{t+2}$ is defined and continuous as long as $t \neq -2$.

So for any $t_0 < -2$ or $t_0 > -2$ and any y_0 $(-\infty < y_0 < \infty)$, the Existence Theorem guarantees a solution exists.

Graphically:



Everywhere except the dark line.

(b) For what points (t_0, y_0) does the Uniqueness Theorem guarantee that there is only one solution satisfying $y(t_0) = y_0$?

$\frac{\partial f}{\partial y} = \frac{\frac{2}{3}y^{-1/3}}{t+2} = \frac{2}{3\cdot\sqrt[3]{y}\cdot(t+2)}$ is defined and continuous as long as $t \neq -2$ and $y \neq 0$.

So for any (t_0, y_0) with $t_0 \neq -2$, $y_0 \neq 0$, the Uniqueness Theorem guarantees that there will be only one solution.

Graphically:

Everywhere except the dark lines