1. (10 pts) Consider the differential equation $ty' - 3y = t^5 - t^2$.

   (a) (2 pts) For what points (t_0, y_0) does the Existence and Uniqueness Theorem guarantee that there exists one and only one solution to this differential equation satisfying the intial condition $y(t_0) = y_0$?

   Standard Form: $y' = \frac{3}{t}y + t^4 - t$

   $$\left.\begin{aligned} F(t,y) &= \frac{3}{t}y + t^4 - t \\ \frac{\partial F}{\partial y} &= \frac{3}{t} \end{aligned}\right\} \text{Both continuous everywhere except at } t=0$$

   $\boxed{\text{Answer: All points } (t_0, y_0) \text{ where } t_0 \neq 0}$

   (b) (4 pts) Find the general solution of the differential equation.

   $y' - \frac{3}{t}y = t^4 - t$ 1st-order linear

   Integrating factor: $u = e^{\int -\frac{3}{t}dt} = e^{-3\ln t} = t^{-3}$

   $\frac{1}{t^3}y' - \frac{3}{t^4}y = t - \frac{1}{t^2}$

   $\frac{1}{t^3}y = \int\left(t - \frac{1}{t^2}\right)dt = \frac{1}{2}t^2 + \frac{1}{t} + C$

   $$\boxed{y = \frac{1}{2}t^5 + t^2 + Ct^3}$$

   (c) (4 pts) How many solutions are there satisfying each of the following initial conditions?

   i. $y(-1) = 0$ 1 (Ex. & Un. Theorem holds here)
   ii. $y(0) = -1$ 0 $\leftarrow$ $y(0) = 0$ for all solutions!
   iii. $y(3) = 1$ 1 (Ex. & Un. Thm. holds)
   iv. $y(0) = 0$ Infinitely many! $\leftarrow$ All solutions satisfy $y(0) = 0$.

2. (10 pts) Solve the initial value problem

$$\underbrace{(6x^2 - \frac{3}{y^2})}_{P}dx + \underbrace{(14\frac{x^3}{y} - 15\frac{x}{y^3})}_{Q}dy = 0, \quad y(2) = 1.$$

$$\frac{\partial P}{\partial y} = \frac{6}{y^3} \qquad \frac{\partial Q}{\partial x} = 42\frac{x^2}{y} - \frac{15}{y^3}$$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, so not exact.

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \frac{21}{y^3} - \frac{42x^2}{y} = \frac{21}{y}\left(\frac{1}{y^2} - 2x^2\right)$$

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{\frac{21}{y}\left(\frac{1}{y^2} - 2x^2\right)}{14\frac{x^3}{y} - 15\frac{x}{y^3}}$$ ... Not a function of only x.

$$\frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P} = \frac{\frac{21}{y}\left(2x^2 - \frac{1}{y^2}\right)}{6x^2 - \frac{3}{y^2}} = \frac{\frac{21}{y}\cancel{\left(2x^2 - \frac{1}{y^2}\right)}}{3\cancel{\left(2x^2 - \frac{1}{y^2}\right)}} = \frac{7}{y} = k(y)!$$

$$u = e^{\int \frac{7}{y}dy} = e^{7\ln y} = y^7$$ ← Integrating factor!

Multiply original equation by y^7:

$$(6x^2y^7 - 3y^5)dx + (14x^3y^6 - 15xy^4)dy = 0$$

(Double-check: $\frac{\partial P}{\partial y} = 42x^2y^6 - 15y^4$, $\frac{\partial Q}{\partial x} = 42x^2y^6 - 15y^4$ ✓)

Potential function: $f(x,y) = \int(6x^2y^7 - 3y^5)dx = 2x^3y^7 - 3xy^5 + C(y)$

$$\frac{\partial f}{\partial y} = \cancel{14x^3y^6} - \cancel{15xy^4} + C'(y) = \cancel{14x^3y^6} - \cancel{15xy^4}$$

So $C'(y) = 0$, so we may let $C = 0$.

$$f(x,y) = 2x^3y^7 - 3xy^5$$

General solution: $2x^3y^7 - 3xy^5 = C$

Plug in $x=2$, $y=1$: $16 - 6 = C \Rightarrow C = 10$

Solution $\boxed{2x^3y^7 - 3xy^5 = 10}$

3. (15 pts) Consider the differential equation

$$y'' + (3 + 2\tan t)y' + (1 + 3\tan t + 2\tan^2 t)y = t\cos t$$

(a) (3 pts) Show that $y_1 = \cos t$ is a solution to the associated homogeneous equation.

$y_1 = \cos t$

$y_1' = -\sin t$

$y_1'' = -\cos t$

Plug in: $(-\cos t) + (3+2\tan t)(-\sin t) + (1+3\tan t + 2\tan^2 t)(\cos t)$

$= \cancel{-\cos t} \cancel{-3\sin t} - \cancel{2\frac{\sin^2 t}{\cos t}} + \cancel{\cos t} + \cancel{3\sin t} + \cancel{2\frac{\sin^2 t}{\cos t}}$

$= 0.$ ✓

(b) (6 pts) Use the method of reduction of order to find a second solution y_2 of the associated homogeneous equation, such that y_2 is linearly independent from y_1.

Assume $y_2 = vy_1 = v\cos t$

$y_2' = v'\cos t - v\sin t$

$y_2'' = v''\cos t - v'\sin t - v'\sin t - v\cos t$

$= v''\cos t - 2v'\sin t - v\cos t$

Plug in:

$(v''\cos t - 2v'\sin t - v\cos t) + (3+2\tan t)(v'\cos t - v\sin t) + (1+3\tan t+2\tan^2 t)v\cos t = 0$

$v''\cos t \cancel{-2v'\sin t} \cancel{-v\cos t} + 3v'\cos t \cancel{+2v'\sin t} \cancel{-3v\sin t} - \cancel{2v\frac{\sin t}{\cos^2 t}} + \cancel{v\cos t} + \cancel{3v\sin t} + \cancel{2v\frac{\sin^2 t}{\cos t}} = 0$

$v''\cos t + 3v'\cos t = 0 \Longrightarrow v'' + 3v' = 0$

$v'' = -3v'$

Let $u = v'$, so $u' = v''$: $u' = -3u \Longrightarrow \frac{u'}{u} = -3$

$\ln u = -3t$

$u = e^{-3t}$

$v' = e^{-3t}$

So $v = \int e^{-3t}dt = -\frac{1}{3}e^{-3t}$

Now $y_2 = v\cos t$

So $\boxed{y_2 = -\frac{1}{3}e^{-3t}\cos t}$

(c) (6 pts) Find the general solution of the full inhomogeneous differential equation.

Note: For the homogeneous solution, constant multiples don't matter, so we can take
$$y_1 = \cos t$$
$$y_2 = e^{-3t}\cos t$$

Wronskian: $W = \det\begin{pmatrix} \cos t & e^{-3t}\cos t \\ -\sin t & -3e^{-3t}\cos t - e^{-3t}\sin t \end{pmatrix}$

$$= -3e^{-3t}\cos^2 t - \cancel{e^{-3t}\sin t\cos t} + \cancel{e^{-3t}\sin t\cos t} = -3e^{-3t}\cos^2 t$$

$g(t) = t\cos t$

$$\int -\frac{y_2 g}{W}\,dt = \int \frac{-\cancel{e^{-3t}}\cancel{\cos t}\cdot t\cancel{\cos t}}{-3\cancel{e^{-3t}}\cancel{\cos^2 t}}\,dt = \frac{1}{3}\int t\,dt = \frac{1}{6}t^2$$

$$\int \frac{y_1 g}{W}\,dt = \int \frac{\cancel{\cos t}\cdot t\cancel{\cos t}}{-3e^{-3t}\cancel{\cos^2 t}}\,dt = -\frac{1}{3}\int te^{3t}\,dt \qquad \begin{matrix} u=t & dv=e^{3t}dt \\ du=dt & v=\frac{1}{3}e^{3t}\end{matrix}$$

$$= -\frac{1}{3}\left(t\cdot\frac{1}{3}e^{3t} - \int \frac{1}{3}e^{3t}\,dt\right) = -\frac{1}{9}te^{3t} + \frac{1}{27}e^{3t}$$
$$= e^{3t}\left(\frac{1}{27} - \frac{1}{9}t\right)$$

Particular solution: $y_p = y_1\int -\frac{y_2 g}{W}\,dt + y_2\int \frac{y_1 g}{W}\,dt$

$$= \cos t\left(\frac{1}{6}t^2\right) + \cancel{e^{-3t}}\cos t\cdot \cancel{e^{3t}}\left(\frac{1}{27} - \frac{1}{9}t\right)$$
$$= \left(\frac{1}{27} - \frac{1}{9}t + \frac{1}{6}t^2\right)\cos t$$

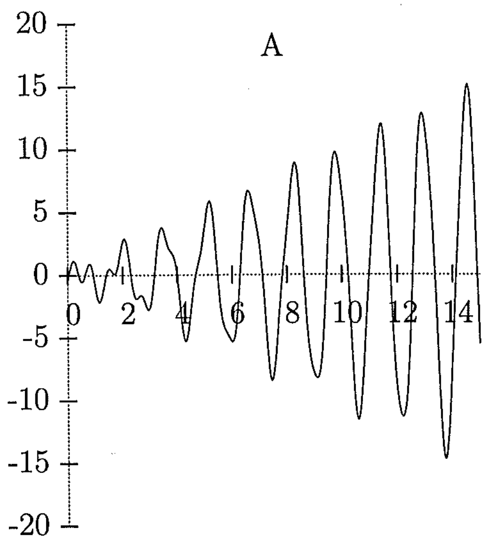
Note: The $\frac{1}{27}\cos t$ term is part of the homogeneous solution, so we can drop it from y_p if we want.

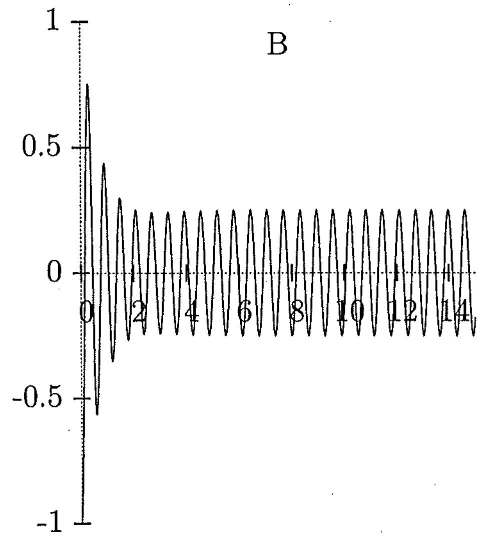
General Solution:
$$\boxed{y = C_1\cos t + C_2 e^{-3t}\cos t + \left(\frac{1}{6}t^2 - \frac{1}{9}t\right)\cos t}$$

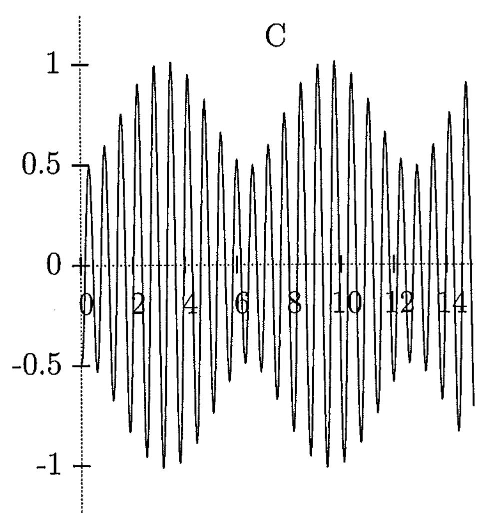
4. (10 pts) Match each of the following five differential equations with the graphs of solutions on the next page, and describe (in one or two words) the type of harmonic motion for each one.
(1 pt each)

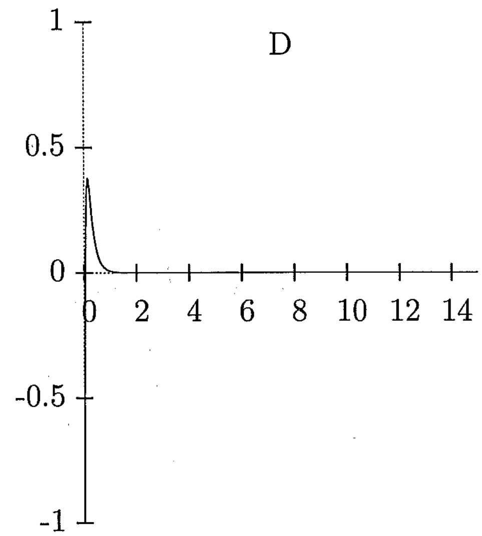
	Linear System	Plot	Type of Harmonic Motion ~~Type of Equilibrium~~
$c=5, \omega_0=10$ $c<\omega_0$	$y''+10y'+100y=0$	E	Underdamped (unforced)
$c=1$	$y''+2y'+100y=5\cos(10t)$	B	Damped & Forced $\Longrightarrow$ Transient plus Steady-State
$c=0$! $\omega_0=10=\omega_F=10$	$y''+100y=5\cos(10t)$	A	Resonance
$c=12.5$ $\omega_0=10$ $c>\omega_0$	$y''+25y'+100y=0$	D	Overdamped (unforced)
$c=0$ $\omega_0=10$ $\omega=9$	$y''+100y=5\cos(9t)$	C	Interference, i.e. "Beats"

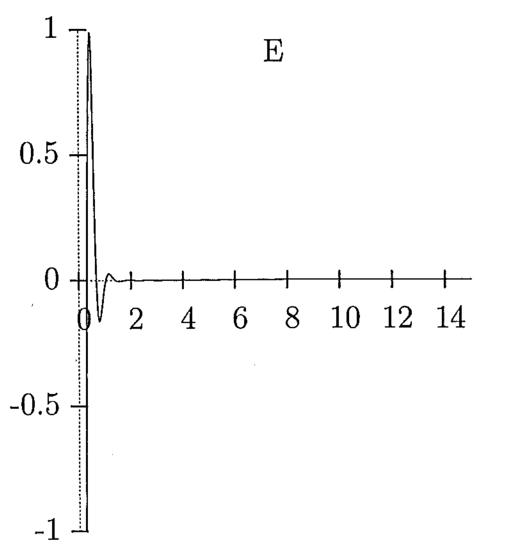
$\omega \neq \omega_0 \Longrightarrow$ interference


A
20
15
10
5
0
-5
-10
-15
-20
0 2 4 6 8 10 12 14

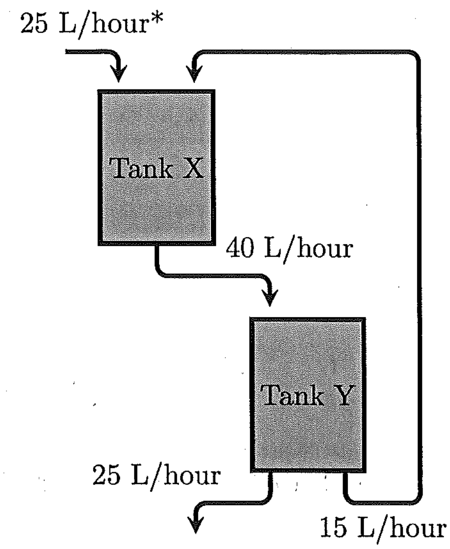

B
1
0.5
0
-0.5
-1
0 2 4 6 8 10 12 14


C
1
0.5
0
-0.5
-1
0 2 4 6 8 10 12 14


D
1
0.5
0
-0.5
-1
0 2 4 6 8 10 12 14


E
1
0.5
0
-0.5
-1
0 2 4 6 8 10 12 14

5. (15 pts) Tank X contains 10 L of a salt-water solution, and tank Y contains 20 L. At time $t = 0$, solution begins flowing into tank X at a rate of 25 L/hour, **with a concentration of* 5 *g/L of salt.* At the same time, a drain is opened to allow the solution to flow out of tank X into tank Y at 40 L/hour; another drain allows the solution in tank Y to flow out at 25 L/hour, and a pump moves the solution from tank Y back into tank X at a rate of 15 L/hour. *(See diagram. Note that the volume of solution in each tank will remain constant.)*



(a) (5 pts) Let x be the amount of salt in tank X and y the amount in tank Y at time t. Set up a system of differential equations for x and y.

$$x' = \text{rate in} - \text{rate out}$$
$$= (25\ \text{L/hr})(5\ \text{g/L}) + (15\ \text{L/hr})\left(\frac{y}{20\ \text{L}}\right) - (40\ \text{L/hr})\left(\frac{x}{10\ \text{L}}\right)$$
$$= 125\ \text{g/hr} + \frac{3}{4}y\ /\text{hr} - 4x\ /\text{hr}$$

$$y' = \text{rate in} - \text{rate out}$$
$$= (40\ \text{L/hr})\left(\frac{x}{10\ \text{L}}\right) - (25 + 15\ \text{L/hr})\left(\frac{y}{20\ \text{L}}\right)$$
$$= 4x\ /\text{hr} - 2y\ /\text{hr}$$

Answer: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -4 & \frac{3}{4} \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 125 \\ 0 \end{pmatrix}$

$$x' = -4x + \frac{3}{4}y + 125$$
$$y' = 4x - 2y$$

(b) (8 pts) Solve the system of equations you found in part (a).

$A = \begin{pmatrix} -4 & 3/4 \\ 4 & -2 \end{pmatrix}$ $\quad (-4-\lambda)(-2-\lambda) - 3 = \lambda^2 + 6\lambda + 5 = (\lambda+5)(\lambda+1) = ($

$\lambda_1 = -5, \quad \lambda_2 = -1$

For $\lambda_1 = -5$: $\begin{pmatrix} 1 & 3/4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \vec{V_1} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \Longrightarrow \vec{y_1} = e^{-5t}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

For $\lambda_2 = -1$: $\begin{pmatrix} -3 & 3/4 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \vec{V_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \Longrightarrow \vec{y_2} = e^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

Fundamental matrix: $Y(t) = \begin{pmatrix} -3e^{-5t} & e^{-t} \\ 4e^{-5t} & 4e^{-t} \end{pmatrix}$

$\det Y(t) = -12e^{-6t} - 4e^{-6t} = -16e^{-6t}$, so $Y(t)^{-1} = \frac{1}{-16e^{-6t}}\begin{pmatrix} 4e^{-t} & -e^{-t} \\ -4e^{-5t} & -3e^{-5t} \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{4}e^{5t} & \frac{1}{16}e^{5t} \\ \frac{1}{4}e^{t} & \frac{3}{16}e^{t} \end{pmatrix}$

$\int Y(t)^{-1}\vec{g}(t)\,dt = \int \begin{pmatrix} -\frac{1}{4}e^{5t} & \frac{1}{16}e^{5t} \\ \frac{1}{4}e^{t} & \frac{3}{16}e^{t} \end{pmatrix}\begin{pmatrix} 125 \\ 0 \end{pmatrix} dt = \frac{125}{4}\int \begin{pmatrix} -e^{5t} \\ e^{t} \end{pmatrix} dt = \begin{pmatrix} -\frac{25}{4}e^{5t} \\ \frac{125}{4}e^{t} \end{pmatrix}$

Particular solution: $\vec{y_p} = \begin{pmatrix} -3e^{-5t} & e^{-t} \\ 4e^{-5t} & 4e^{-t} \end{pmatrix}\begin{pmatrix} -\frac{25}{4}e^{5t} \\ \frac{125}{4}e^{t} \end{pmatrix} = \begin{pmatrix} 50 \\ 100 \end{pmatrix}$

General Solution: $\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-5t}\begin{pmatrix} -3 \\ 4 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 50 \\ 100 \end{pmatrix}}$

or $\boxed{\begin{cases} x = -3C_1 e^{-5t} + C_2 e^{-t} + 50 \\ y = 4C_1 e^{-5t} + 4C_2 e^{-t} + 100 \end{cases}}$

(c) (2 pts) Using your answer from part (b), or other means, what can you say about the amount of salt in each tank as $t \to \infty$?

As $t \to \infty$, $x \to 50g$ and $y \to 100g$.

That is, the amount (mass) of salt in tank X asymptotically approaches 50g, and the amount in tank Y approaches 100g.

Note that there is only one input to the system, and one output. So it is reasonable to guess that eventually all of the solution in the system will approach the same concentration as the input: 5 g/L. Multiplying this concentration by the volumes gives the same result.

6. (10 pts) Consider the inhomogeneous linear system of differential equations

$$\mathbf{y}' = \begin{pmatrix} 0 & \sec t \\ \sec t & 0 \end{pmatrix} \mathbf{y} + \begin{pmatrix} \cos t \\ \sec t \end{pmatrix}.$$

(a) (3 pts) Show that $\mathbf{y}_1 = \begin{pmatrix} \sec t \\ \tan t \end{pmatrix}$ and $\mathbf{y}_2 = \begin{pmatrix} \tan t \\ \sec t \end{pmatrix}$ are solutions of the associated homogeneous system.

$$\vec{y}_1' = \begin{pmatrix} \sec t \tan t \\ \sec^2 t \end{pmatrix} \quad \checkmark$$

$$A\vec{y} = \begin{pmatrix} 0 & \sec t \\ \sec t & 0 \end{pmatrix}\begin{pmatrix} \sec t \\ \tan t \end{pmatrix} = \begin{pmatrix} \sec t \tan t \\ \sec^2 t \end{pmatrix}$$

$$\vec{y}_2' = \begin{pmatrix} \sec^2 t \\ \sec t \tan t \end{pmatrix} \quad \checkmark$$

$$A\vec{y} = \begin{pmatrix} 0 & \sec t \\ \sec t & 0 \end{pmatrix}\begin{pmatrix} \tan t \\ \sec t \end{pmatrix} = \begin{pmatrix} \sec^2 t \\ \sec t \tan t \end{pmatrix}$$

(b) (7 pts) Find the general solution of the inhomogeneous system.
(Note: The trig. identity $\sec^2 t - \tan^2 t = 1$ will be useful here.)

Fundamental Matrix: $Y(t) = \begin{pmatrix} \sec t & \tan t \\ \tan t & \sec t \end{pmatrix}$

$$\det Y(t) = \sec^2 t - \tan^2 t = 1$$

$$Y(t)^{-1} = \frac{1}{\det Y(t)} \begin{pmatrix} \sec t & -\tan t \\ -\tan t & \sec t \end{pmatrix} = \begin{pmatrix} \sec t & -\tan t \\ -\tan t & \sec t \end{pmatrix}$$

$$Y(t)^{-1}\vec{g}(t) = \begin{pmatrix} \sec t & -\tan t \\ -\tan t & \sec t \end{pmatrix}\begin{pmatrix} \cos t \\ \sec t \end{pmatrix} = \begin{pmatrix} 1 - \sec t \tan t \\ -\sin t + \sec^2 t \end{pmatrix}$$

$$\int (1 - \sec t \tan t)\,dt = t - \sec t$$

$$\int (-\sin t + \sec^2 t)\,dt = \cos t + \tan t$$

$$\vec{y}_p = Y(t)\int Y(t)^{-1}\vec{g}(t)\,dt = \begin{pmatrix} \sec t & \tan t \\ \tan t & \sec t \end{pmatrix}\begin{pmatrix} t - \sec t \\ \cos t + \tan t \end{pmatrix}$$

$$= \begin{pmatrix} t\sec t - \sec^2 t + \sin t + \tan^2 t \\ t\tan t - \cancel{\sec t \tan t} + 1 + \cancel{\sec t \tan t} \end{pmatrix} = \begin{pmatrix} t\sec t - 1 + \sin t \\ t\tan t + 1 \end{pmatrix}$$

$$\boxed{\text{General solution: } \vec{y} = C_1\begin{pmatrix} \sec t \\ \tan t \end{pmatrix} + C_2\begin{pmatrix} \tan t \\ \sec t \end{pmatrix} + \begin{pmatrix} t\sec t - 1 + \sin t \\ t\tan t + 1 \end{pmatrix}}$$

7. (10 pts) Compute the general solution of the system of equations

$$\mathbf{y}' = \begin{pmatrix} 8 & -7 & -9 \\ 0 & -2 & 0 \\ 5 & 4 & -4 \end{pmatrix} \mathbf{y}.$$

(Hint: To find the characteristic polynomial, it will be easiest to expand along the second row.)

$$\det\begin{pmatrix} 8-\lambda & -7 & -9 \\ 0 & -2-\lambda & 0 \\ 5 & 4 & -4-\lambda \end{pmatrix} = (-2-\lambda)\cdot\det\begin{pmatrix} 8-\lambda & -9 \\ 5 & -4-\lambda \end{pmatrix} = (-2-\lambda)\left[(8-\lambda)(-4-\lambda)+45\right]$$

$$= -(\lambda+2)(\lambda^2-4\lambda+13)=0 \qquad \lambda=-2 \qquad \lambda=\frac{4\pm\sqrt{16-52}}{2} = 2\pm 3i$$

For $\lambda=-2$: $\begin{pmatrix} 10 & -7 & -9 \\ 0 & 0 & 0 \\ 5 & 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$

$$10v_1-7v_2-9v_3=0$$
$$5v_1+4v_2-2v_3=0$$
$$-15v_2-5v_3=0$$
$$v_3=-3v_2$$

$$5v_1+4v_2+6v_2=0$$
$$v_1=-2v_2$$

$$\vec{v}=\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Solution: $\vec{y}_1 = e^{-2t}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

For $\lambda=2+3i$: $\begin{pmatrix} 6-3i & -7 & -9 \\ 0 & -4-3i & 0 \\ 5 & 4 & -6-3i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow$ $v_2=0$

$$(6-3i)v_1-9v_3=0$$

$$\vec{v}=\begin{pmatrix} 3 \\ 0 \\ 2-i \end{pmatrix}$$

Solutions: $\vec{y} = e^{(2+3i)t}\begin{pmatrix} 3 \\ 0 \\ 2-i \end{pmatrix}$

$$= e^{2t}(\cos 3t + i\sin 3t)\begin{pmatrix} 3 \\ 0 \\ 2-i \end{pmatrix} = e^{2t}\begin{pmatrix} 3\cos 3t + i\cdot 3\sin 3t \\ 0 \\ 2\cos 3t+\sin 3t + i(2\sin 3t-\cos 3t) \end{pmatrix}$$

$$\vec{y}_2 = \begin{pmatrix} 3\cos 3t \\ 0 \\ 2\cos 3t+\sin 3t \end{pmatrix} \qquad \vec{y}_3 = \begin{pmatrix} 3\sin 3t \\ 0 \\ 2\sin 3t-\cos 3t \end{pmatrix}$$

$$\boxed{\text{General solution: } \vec{y} = C_1\begin{pmatrix} 2e^{-2t} \\ -e^{-2t} \\ 3e^{-2t} \end{pmatrix} + C_2\begin{pmatrix} 3\cos 3t \\ 0 \\ 2\cos 3t+\sin 3t \end{pmatrix} + C_3\begin{pmatrix} 3\sin 3t \\ 0 \\ 2\sin 3t-\cos 3t \end{pmatrix}}$$

8. (10 pts) Let $A = \begin{pmatrix} 6 & 1 \\ -4 & 2 \end{pmatrix}$. Compute e^{At}.

$$\det\begin{pmatrix} 6-\lambda & 1 \\ -4 & 2-\lambda \end{pmatrix} = (6-\lambda)(2-\lambda)+4 = \lambda^2-8\lambda+16 = (\lambda+4)^2 = 0$$

$\lambda = 4$ (algebraic multiplicity 2)

Eigenvectors: $\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ So the geometric multiplicity is 1.

Solution: $\vec{y}_1 = e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Generalized eigenvector:

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad 2w_1 + w_2 = 1 \qquad \vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution: $\vec{y}_2 = e^{4t}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}t\right)$

$$= e^{4t}\begin{pmatrix} t \\ 1-2t \end{pmatrix}$$

Fundamental matrix: $Y(t) = \begin{pmatrix} e^{4t} & te^{4t} \\ -2e^{4t} & (1-2t)e^{4t} \end{pmatrix}$

$$Y(0) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \qquad Y(0)^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

So $e^{At} = Y(t)Y(0)^{-1} = \begin{pmatrix} e^{4t} & te^{4t} \\ -2e^{4t} & (1-2t)e^{4t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$$= \boxed{\begin{pmatrix} (1+2t)e^{4t} & te^{4t} \\ -4te^{4t} & (1-2t)e^{4t} \end{pmatrix}}$$

9. (10 pts) For the nonlinear system of differential equations given below, plot the nullclines on the axes provided, and draw arrows along them in the correct directions. Sketch arrows to fill out the rest of the phase portrait. Find all equilibrium points and classify each one.

(Hint: There are seven equilibria, but you should only really need to use the Jacobian to classify a few of them (one, or maybe three). The rest should be obvious if you do the nullclines correctly.)

$\frac{\partial f}{\partial x} = -(x^2-4) + 2x(2y-x) = 4 - 3x^2 + 4xy$

$$\begin{cases} x' = (2y - x)(x^2 - 4) \\ y' = -x(y^2 - 4) \end{cases}$$

Equilibrium	Type	
$(0,0)$	Spiral Source	From Jacobian
$(2,2)$	Saddle	Obvious from nullclines
$(-2,2)$	Saddle	
$(-2,-2)$	Saddle	
$(2,-2)$	Saddle	
$(4,2)$	Nodal Sink	From phase portrait, or check Jacobian
$(-4,-2)$	Nodal Sink	

x-nullcline ("vertical" nullcline)

$(2y-x)(x^2-4)=0$

$(x=2$ or $x=-2$ or $y=\frac{1}{2}x)$

y-nullcline ("horizontal" nullcline)

$-x(y^2-4)=0$

$(y=2$ or $y=-2$ or $x=0)$

Jacobian:

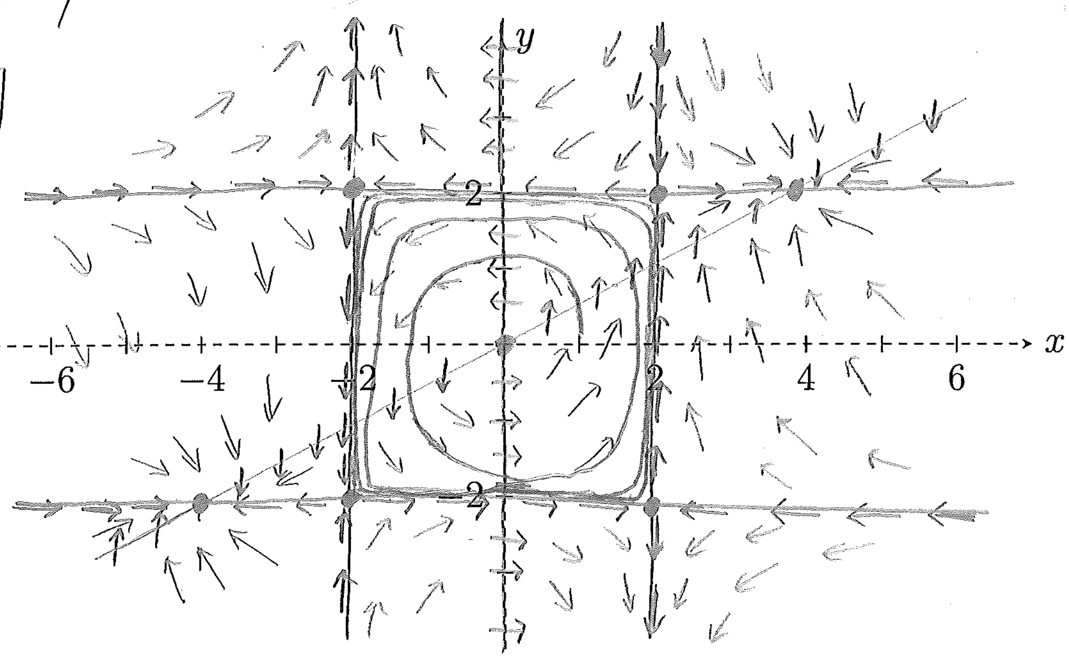
$$\begin{pmatrix} 4-3x^2+4xy & 2(x^2-4) \\ 4-y^2 & -2xy \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 4 & -8 \\ 4 & 0 \end{pmatrix}$$

$T=4$, $D=32$

$T^2-4D<0$

Spiral Source



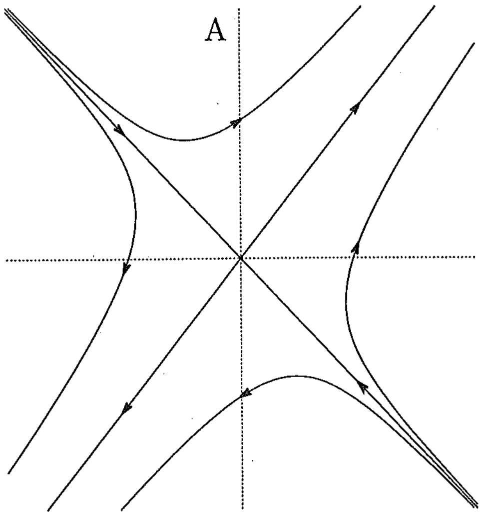
Bonus: (3pts) Describe the long-term behavior of the solution curve that starts at the point $(1,0)$. Note: By the uniqueness theorem, the solution can't cross the other solution lines at $y=\pm 2$ and $x=\pm 2$.

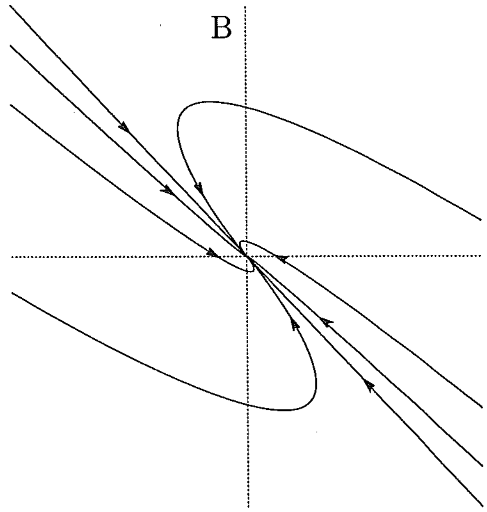
So the curve will "spiral" outward until it asymptotically approaches the square formed by those other curves. It will then spend eternity bouncing →

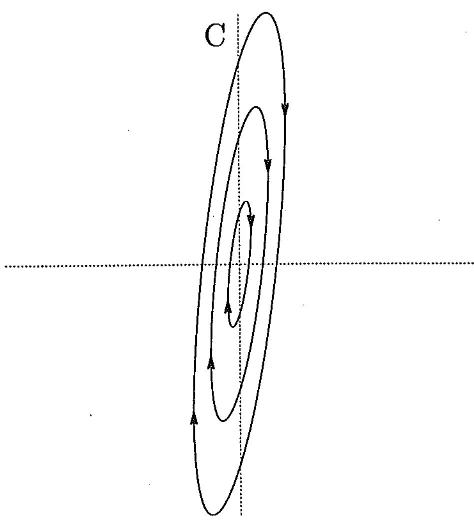
(cont'd) → between the four saddle points in a cycle.

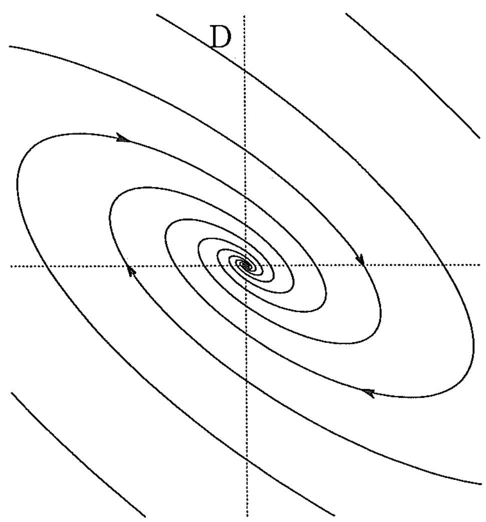
10. (10 pts) Match each of the following five systems of differential equations with the phase plane graphs on the next page, and classify the type of equilibrium at the origin for each one.
(1 pt each)

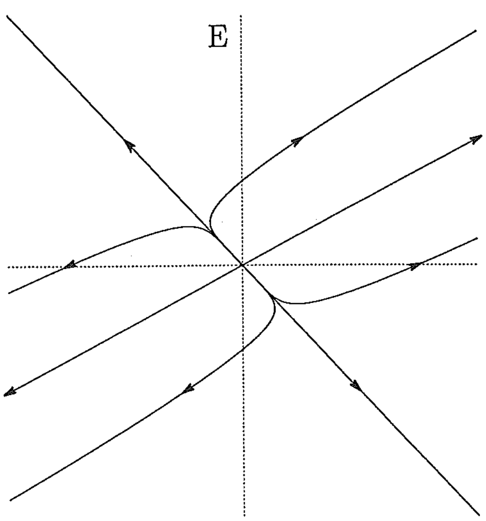
	Linear System	Plot	Type of Equilibrium
$T=-2,\ D=10$ $T^2-4D<0$	$\mathbf{y}' = \begin{pmatrix} 2 & 6 \\ -3 & -4 \end{pmatrix} \mathbf{y}$	D	Spiral Sink
$T=8,\ D=7$ $T^2-4D>0$	$\mathbf{y}' = \begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix} \mathbf{y}$	E	Nodal Source
$T=4,\ D=4$ $T^2-4D=0$	$\mathbf{y}' = \begin{pmatrix} -2 & -4 \\ 4 & 6 \end{pmatrix} \mathbf{y}$	B	Degenerate Source
$T=5,\ D=-24$	$\mathbf{y}' = \begin{pmatrix} 2 & 5 \\ 6 & 3 \end{pmatrix} \mathbf{y}$	A	Saddle
$T=0,\ D=11$ $T^2-4D<0$	$\mathbf{y}' = \begin{pmatrix} -3 & 5 \\ -4 & 3 \end{pmatrix} \mathbf{y}$	C	Center


A


B


C


D


E