Midterm 2

Last Name: [redacted]

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Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above. **You may not use calculators**, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use any available space on the exam for scratch work, including the backs of pages. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Problem	Score
1	10 /10
2	8 /10
3	10 /10
4	10 /10
Total:	38 /40

1) Solve the initial value problem.

$$x'' + 7x' - 18x = 0, \quad x(0) = 1, x'(0) = 13$$

Homog → $\lambda^2 + 7\lambda - 18 = 0$

$(\lambda+9)(\lambda-2) = 0$

$\lambda = -9, \lambda = 2$

$x_c(t) = C_1 e^{-9t} + C_2 e^{2t}$

$x(0) = 1 = C_1 + C_2$

$x'(t) = -9C_1 e^{-9t} + 2C_2 e^{2t}$

$x'(0) = 13 = -9C_1 + 2C_2$

$C_1 + C_2 = 1$

$-9C_1 + 2C_2 = 13$

$11C_2 = 22$

$C_2 = 2$

$C_1 = -1$

$\boxed{x(t) = -e^{-9t} + 2e^{2t}}$

5

Check

$x(0) = -1 + 2 = 1$

$x'(t) = 9e^{-9t} + 4e^{2t}$

$x'(0) = 9 + 4 = 13$

$x''(t) = -81e^{-9t} + 8e^{2t}$

$-81e^{-9t} + 8e^{2t}$

$+63e^{-9t} + 28e^{2t}$

$+18e^{-9t} - 36e^{2t}$

$= 0$

2) A 2 kg mass is attached to a horizontal spring with spring constant 20 N/m. The damping constant is 4 kg/s. Let x(t) denote the position of the mass (in meters) after t seconds. (Here x = 0 m corresponds to the equilibrium position of the spring.) Suppose at time t = 0 the position of the mass is 0 and the velocity is 6 m/s. Find the position of the mass at time t = π/6 s.

$m = 2kg$ $\quad x(0)=0$

$K = 20 N/m$ $\quad x'(0)=6$

$\mu = 4 kg/s$ $\quad$ what is $x(\frac{\pi}{6})$

$mx'' = -\mu x' - Kx$

$x'' + \frac{\mu}{m}x' + \frac{K}{m}x = 0$

$\lambda = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$

$x'' + 2x' + 10x = 0$

$\lambda^2 + 2\lambda + 10 = 0$

$(\lambda+1)^2 = -9$

$\lambda = -1 \pm 3i$ ✓

$x(t) = e^{-t}[C_1\cos(3t) + C_2\sin(3t)]$

$x(0) = 0 = C_1$ ✓

$x'(t) = -e^{-t}[C_1\cos(3t) + C_2\sin(3t)] + e^{-t}[-3C_1\sin(3t) + 3C_2\cos(3t)]$

$x'(0) = 6 = -C_1 + 3C_2 \rightarrow 6 = 3C_2$ ✓

$C_2 = 2$

$x(t) = e^{-t}2\sin(3t)$

$x(\frac{\pi}{6}) = e^{-\frac{\pi}{6}}\sin(\frac{\pi}{2})$

$\boxed{x(\frac{\pi}{6}) = 2e^{-\frac{\pi}{6}}}$

4

Check

$x'(t) = -e^{-t}2\sin(3t) + e^{-t}6\cos(3$

$x'(0) = 6$

3a) Find a basis, β, of eigenvectors for the matrix A.

$$A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}, \qquad \beta = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$$\det(A - \lambda Id) = \det \begin{bmatrix} 3-\lambda & 4 \\ -2 & -3-\lambda \end{bmatrix} = -9 + \lambda^2 + 8 = 0$$

$$\lambda^2 - 1 = 0$$

$$(\lambda+1)(\lambda-1) = 0$$

$$\lambda = 1, \lambda = -1$$

For $\lambda = 1$

$$A - Id = \begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad x = -2y \quad u_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For $\lambda = -1$

$$A + Id = \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad x = -y \quad u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\boxed{\beta = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}}$$ ✓ ✓

span of these 2 vectors

5

Chk

$$_\gamma[Id]_\beta \, _\beta[Id]_\gamma = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$_\beta[Id]_\gamma \, A \, _\gamma[Id]_\beta = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \checkmark$$

b) Let γ denote the standard basis. Compute the following matrices. (Hint: The computations should be very short!)

$_\gamma[Id]_\beta = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$ ✓ $\qquad$ $_\beta[Id]_\gamma = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$ ✓ $\qquad$ $_\beta[A]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ✓

$$\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= -1 \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

5

4) Find the general solution to the differential equation.

$$x'' - 2x' + x = 4e^t$$

Homog: $x'' - 2x' + x = 0$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1$

$y_h(t) = C_1 e^t + C_2 t e^t$

$\boxed{y(t) = C_1 e^t + C_2 t e^t + 2t^2 e^t}$

5

$y_p = Ae^t$

$y_p' = Ae^t$ $\quad x'' - 2x' + x = 0 = 4e^t$

$y_p'' = Ae^t$

$\downarrow$

$y_p = Ate^t$

$y_p' = Ae^t + Ate^t$

$y_p'' = 2Ae^t + Ate^t$

$$\left.\begin{array}{l} 2Ae^t + Ate^t \\ -2Ae^t - 2Ate^t \\ \quad + Ate^t \end{array}\right\} = 0$$

$\downarrow$

$y_p = At^2 e^t$

$y_p' = At^2 e^t + 2Ate^t$

$y_p'' = At^2 e^t + 4Ate^t + 2Ae^t$

$$\begin{array}{l} \cancel{At^2e^t} + \cancel{4Ate^t} + 2Ae^t = 4e^t \\ -\cancel{2At^2e^t} - \cancel{4Ate^4} \\ +\cancel{At^2e^t} \end{array}$$

$A = 2$

$y_p(t) = 2t^2 e^t$

Check

$y(t) = C_1 e^t + C_2 t e^t + 2t^2 e^t$

$y'(t) = C_1 e^t + C_2 e^t + C_2 t e^t + 2t^2 e^t + 4te^t$

$y''(t) = C_1 e^t + 2C_2 e^t + C_2 t e^t + 2t^2 e^t + 8te^t + 4e^t$