Midterm 1

Last Name: Key

First Name: ______________________

Student ID: ______________________

Signature: ______________________

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above. **You may not use calculators**, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use any available space on the exam for scratch work, including the backs of pages. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Problem	**Score**
1	**/5**
2	**/5**
3	**/5**
4	**/5**
Total:	**/20**

1) Solve the differential equation. If possible, find an explicit solution for y as a function of x. Otherwise, find an implicit solution. (Hint: Suppose there is an integrating factor which is a function of y alone.)

$u = u(y)$

$$xy^2dx + (1 + 3x^2y)dy = 0$$

$$u\,xy^2dx + u(1+3x^2y)dy = 0$$

Closedness: $u'(y)xy^2 + 2uxy = 6uxy$

Solve for $u(y)$: $u'xy^2 = 4uxy$

$$u' = \frac{4u}{y}$$

$$\int \frac{du}{u} = \int \frac{4}{y}\,dy$$

$$\ln|u| = 4\ln|y| = \ln(y^4)$$

$$u = y^4$$

Solve exact equation: $\underbrace{xy^6}_{\frac{\partial F}{\partial x}}\,dx + \underbrace{(y^4+3x^2y^5)}_{\frac{\partial F}{\partial y}}dy = 0$

$$F(x,y) = \int \frac{\partial F}{\partial x}\,dx = \frac{1}{2}x^2y^6 + c(y)$$

$$c'(y) + 3x^2y^5 = \frac{\partial F}{\partial y} = y^4 + 3x^2y^5$$

$$c'(y) = y^4 \Rightarrow c(y) = \frac{1}{5}y^5$$

$$F(x,y) = \frac{1}{2}x^2y^6 + \frac{1}{5}y^5$$

$$\boxed{\text{Solution: } \frac{1}{2}x^2y^6 + \frac{1}{5}y^5 = c}$$

2) A 20˚C can of soda is placed inside an empty freezer. When the freezer is opened at t = 0 hours, the incoming warm air raises the temperature inside to 5˚C. Once closed, the air temperature inside the freezer (in ˚C) after t hours is given by $5e^{-2\ln(2)t}$. The constant of proportionality, k, in Newton's Law of Cooling is given by ln(2). Find the temperature of the soda after 1 hour. **Please express your answer as a rational number.**

$$\frac{dT}{dt} = -k(T - A)$$

$$T' = -\ln(2)(T - 5e^{-2\ln(2)t})$$

$$T' + \ln(2)T = 5\ln(2)e^{-2\ln(2)t}$$

Integrating factor u:

$u' = \ln(2)u$

$\Rightarrow u = e^{\ln(2)t}$

$$uT' + \ln(2)uT = 5\ln(2)ue^{-2\ln(2)t}$$

$$(uT)' = 5\ln(2)ue^{-2\ln(2)t}$$

$$e^{\ln(2)t}T = 5\ln(2)\int e^{-\ln(2)t}\,dt$$

$$e^{\ln(2)t}T = -5e^{-\ln(2)t} + C$$

$$T = -5e^{-2\ln(2)t} + C e^{-\ln(2)t}$$

$$20 = T(0) = -5 + C$$

$$25 = C$$

$$T(t) = -5e^{-2\ln(2)t} + 25e^{-\ln(2)t}$$

$$T(1) = -5e^{-2\ln(2)} + 25e^{-\ln(2)}$$

$$= -5e^{\ln(\frac{1}{4})} + 25e^{\ln(\frac{1}{2})}$$

$$= -5(\tfrac{1}{4}) + 25(\tfrac{1}{2})$$

$$= -\frac{5}{4} + \frac{25}{2}$$

$$= \frac{45}{4}\ ^\circ C$$

3) Solve the initial value problem. If possible, find an explicit solution for y as a function of x. Otherwise, find an implicit solution.

$$(y^2 + 3xy)dx + (xy + x^2)dy = 0, \quad y(1) = 1$$

Homogeneous of degree 2: $y = vx$

$$(v^2x^2 + 3vx^2)dx + (vx^2 + x^2)(x\,dv + v\,dx) = 0$$

$$(v^2x^2 + 3vx^2 + v^2x^2 + vx^2)\,dx + (vx^3 + x^3)\,dv = 0$$

$$(2v^2x^2 + 4vx^2)\,dx + (vx^3 + x^3)\,dv = 0$$

$$x^3(v+1)\,dv = -x^2(2v^2 + 4v)\,dx$$

$$\frac{v+1}{2(v^2+2v)}\,dv = -\frac{1}{x}\,dx$$

$u = v^2 + 2v$

$du = 2v + 2\,dv$

$\frac{du}{2(v+1)} = dv$

$$\int \frac{v+1}{v^2+2v}\,dv = -\int \frac{2}{x}\,dx$$

$$\frac{1}{2}\int \frac{1}{u}\,du = -2\ln|x| + c$$

$$\ln|u| = -4\ln|x| + C =$$

$$\ln|v^2 + 2v| = \ln\left(\frac{1}{x^4}\right) + c$$

$v = \frac{y}{x}$

$$v^2 + 2v = \frac{A}{x^4}$$

$$\frac{y^2}{x^2} + \frac{2y}{x} - \frac{A}{x^4} = 0$$

$y(1) = 1 \Rightarrow \quad 1 + 2 - A = 0 \Rightarrow A = 3$

$$\frac{y^2}{x^2} + \frac{2y}{x} - \frac{3}{x^4} = 0$$

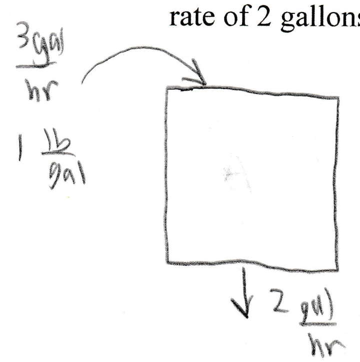
$$y^2 + 2xy - \frac{3}{x^2} = 0$$

$$y = \frac{-2x + \sqrt{4x^2 + \frac{12}{x^2}}}{2}$$

$$\boxed{y = -x + \sqrt{x^2 + 3/x^2}}$$

positive solution yields $y(1) = 1$

4) A large tank, capable of holding 100 gallons of water, currently holds 1 gallon of saltwater solution, at a concentration of 2 pounds per gallon. At time t = 0, saltwater solution at a concentration of 1 pound per gallon pours into the tank at a rate of 3 gallons per hour. At the same time, a small drain at the bottom of the tank is opened, allowing water to leave the tank at a rate of 2 gallons per hour. What is the mass of the salt in the tank after two hours?



$V(t) = 1+t$

$C(0) = 2 \text{ lb/gal}$

$m(0) = 2 \text{ lb}$

$m' = m_{in}' - m_{out}'$

$m' = \left(3 \frac{gal}{hr}\right)\left(1 \frac{lb}{gal}\right) - 2\frac{gal}{hr}\left(\frac{m}{V}\right)$

$m' = 3 - \frac{2m}{1+t}$

$m_h' + \frac{2m_h}{1+t} = 0$

$m_h' = -\frac{2m_h}{1+t}$

$\int \frac{dm_h}{m_h} = -\int \frac{2}{1+t}\,dt$

$\ln|m_h| = -2\ln|1+t|$

$m_h = \frac{1}{(1+t)^2}$

$m' + \frac{2m}{1+t} = 3$

$m = v m_h \Rightarrow v' m_h + \underbrace{\cancel{v m_h' + \frac{2m_h}{1+t} v}}_{0} = 3$

$v' = \frac{3}{m_h} = 3(1+t)^2$

$v = (1+t)^3 + C$

$m(t) = v m_h = 1+t + \frac{C}{(1+t)^2}$

$2 = m(0) = 1 + C$

$1 = C$

$m(t) = 1+t+\frac{1}{(1+t)^2}$

$m(2) = 3 + \frac{1}{3^2}$

$m(2) = 3 + \frac{1}{9}$

$\boxed{m(2) = \frac{28}{9} \text{ lb}}$