Math 33B
Fall 2014
Midterm1
10/31/14
Time Limit: 50 Minutes

Name (Print): [redacted]

[redacted]

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem. **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit. Formula that you might or might not need:

- The differential form $Pdx + Qdy$ has an integrating factor depending on one of the variables under the following conditions.
  If $h = \frac{1}{Q}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$ is a function of x only, then $\mu = e^{\int h(x)dx}$ is an integrating factor.
  If $g = \frac{1}{P}(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$ is a function of y only, then $\mu = e^{-\int g(x)dx}$ is an integrating factor.

- $e^{i\theta} = \cos\theta + i\sin\theta$

Do not write in the table to the right.

Problem	Points	Score
1	15	15
2	15	15
3	15	13
4	15	13
5	20	20
6	20	20
Total:	100	96

[redacted]

D.C.

1. (15 points) Solve the equation $y' = \frac{x^2}{\sin y}$.

15

$\sin.(y) \cdot y' = x^2$

$\sin.(y) \cdot dy = x^2 \cdot dx$

$\int (\sin.(y)) \cdot dy = \int (x^2) \cdot dx$

$-\cos.(y) = \frac{1}{3} x^3 + C$

$\cos.(y) = \frac{-1}{3} x^3 + C$

$y = \pm \cos.^{-1}\left(\frac{-1}{3} x^3 + C \right)$

D.C.

2. (15 points) Solve the equation $e^y dx + (xe^y + y)dy = 0$.

$F(x,y) = x \cdot e^y + \phi(y)$

$F(x,y) = x \cdot e^y + \frac{1}{2}y^2 + \phi(x)$

We see $F(x,y) = x \cdot e^y + \frac{1}{2}y^2$ satisfies both.

$\boxed{x \cdot e^y + \frac{1}{2}y^2 = C}$

15

D.C.

3. (15 points) Solve the equation $y'' + 6y' + 9y = 0$, $y(0) = 0$, $y'(0) = 1$.

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda + 3)^2 = 0$

$\lambda = -3$, multiplicity 2.

$y(t) = C_1 \cdot e^{(-3t)} + C_2 t \cdot e^{(-3t)}$ +10

$y(0) = C_1 + \cancel{C_2} = 0$

$y'(0) = -3C_1 + C_2(0+1)$

$= -3C_1 + C_2 = 1$

$C_1 = -C_2$

$3C_2 + C_2 = 1$

$4C_2 = 1$

$C_2 = \frac{1}{4} \Rightarrow C_1 = \frac{-1}{4}$

$$\boxed{y(t) = \frac{-1}{4} \cdot e^{(-3t)} + \frac{1}{4} t \cdot e^{(-3t)}}$$

+13

4. (15 points) Find all the real values of the constant p such that any solution to the differential equation $y'' + py' + 9y = 0$ satisfies $\lim_{t\to+\infty} y(t) = 0$. Justify your answer.

13

$\lambda^2 + \lambda p + 9 = 0$

$$\lambda = \frac{-p \pm \sqrt{p^2 - 36}}{2}$$

$p \in (0,6)$ works, since the ansor is $e^{(at)} \cdot (C_1\cos.(bt) + C_2\sin.(bt))$ and since $a = -p < 0$, the function approaches 0 for $t \to \infty$.

$p = 6$ yields $y(t) = C_1 e^{(-3t)} + C_2 \cdot t e^{(-3t)}$, and $\lim_{t\to\infty} \frac{t}{e^{(3t)}} = \lim_{t\to\infty} \frac{1}{3e^{(3t)}} = 0$, so it works.

For $p^2 - 36 > 0$, we need $-p + \sqrt{p^2-36} < 0$. This gurantees $\lambda_1, \lambda_2 < 0$.

$\sqrt{p^2-36} < p$

$p^2 - 36 < p^2$

$-36 < 0$

$p<0$ case?

But! $\sqrt{p^2-36} < p$ only works for (and always works for) p that is $+$.

So,

$p \in (0,6] \cup [6,\infty)$

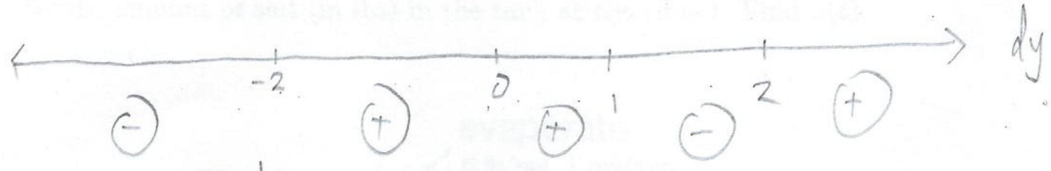
$\boxed{p \in (0,\infty)}$

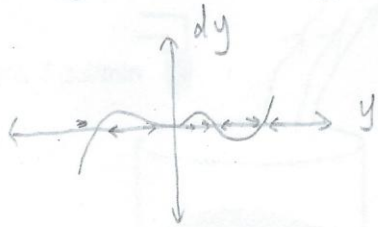
D.C.

5. $dy - 3y^{\frac{2}{3}}(y-1)(y^2-4)dt = 0$

(a) (10 points) Find all the constant solutions that are (asymptotically) stable.

$dy = 3y^{(2/3)}(y-1)(y-2)(y+2)$





$\boxed{y = 1}$

(b) (10 points) Are the solutions to the above equation together with the initial condition $y(0) = 1$ unique? Justify your answer.

dy is continuous.

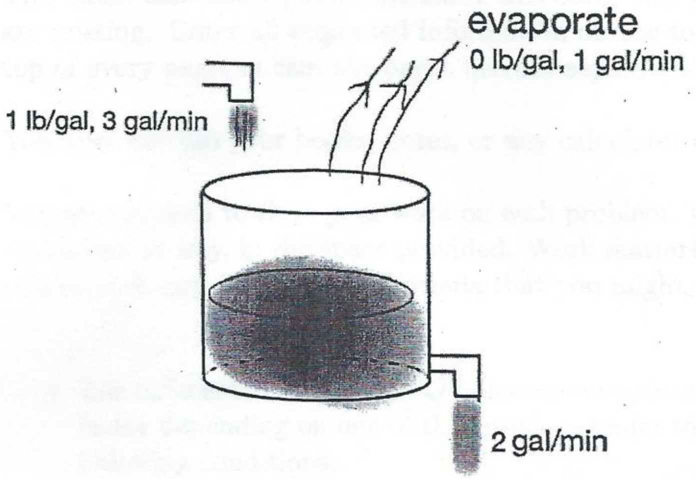
$dy = F(y) = 3y^{(2/3)}(y-1)(y^2-4) = 3y^{(2/3)}(y^3-y^2-4y+4)$

$F'(y) = 2y^{(-1/3)}(y^3-y^2-4y+4) + 3y^{(2/3)}(3y^2-2y-4)$

Is well-defined & ~~continuous~~ around (0,1).

So $\boxed{\text{unique.}}$

6. (20 points) The 10-gal tank is originally filled with 2-gal of salt solution with concentration 1/2 lb/gal. A spigot is opened above the tank and a salt solution with the concentration 1 lb/gal begins flowing into the tank at a rate of 3-gal/min. Meanwhile, a drain is opened at the bottom of the tank allowing the solution to leave the tank at a rate of 3-gal/min. Simultaneously, pure water is evaporating from the solution in the tank into the air at a rate of 2-gal/min. Let $x(t)$ be the amount of salt (in lbs) in the tank at the time t. Find $x(t)$.



$$x'(t) = 1 \cdot 3 - 2 \cdot \frac{x(t)}{2+(3-2-1)t}$$

$$x'(t) = 3 - 2\,\frac{x(t)}{2} = 3 - x(t)$$

$$x'(t) + x(t) = 3$$

$$e^t \cdot x'(t) + e^t \cdot x(t) = 3e^t$$

$$\frac{d}{dt}\left[e^t \cdot x(t)\right] = 3e^t$$

$$e^t \cdot x(t) = 3e^t + C$$

$$x(t) = 3 + \frac{C}{e^t}$$

$$x(0) = 2 \cdot \tfrac{1}{2} = 1 = 3 + \frac{C}{e^t}$$

$$-2 = \frac{C}{e^t} = \frac{C}{e^0} = C$$

$$\boxed{x(t) = 3 - \frac{2}{e^t}}$$