2 (15 points) Solve the equation $e^y dx + (xe^y + y)dy = 0$.

$\frac{\partial M}{\partial y} = e^y$ $\qquad$ $\frac{\partial N}{\partial x} = e^y$

$\Rightarrow$ is exact

$f = \int e^y dx$ $\qquad$ $\frac{\partial f}{\partial y} = xe^y + y = xe^y + \psi'(y)$

$= xe^y + \psi(y)$

$\psi'(y) = y$

$\psi(y) = \frac{1}{2}y^2 + c$

$f = xe^y + \frac{1}{2}y^2 + c = 0$ (b/c exact)

$\boxed{xe^y + \frac{1}{2}y^2 = C}$ $\qquad$ -2

3. (15 points) Solve the equation $y'' + 6y' + 9y = 0$, $y(0) = 0$, $y'(0) = 1$.

characteristic equation:

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda + 3)^2 = 0$

$\lambda_1 = \lambda_2 = \lambda = -3$

e^{-3t} and te^{-3t} are solutions

$y = c_1 e^{-3t} + c_2 t e^{-3t}$

$y' = c_1 e^{-3t}(-3) + c_2 t e^{-3t}(-3) + e^{-3t}(c_2)$

$= -3c_1 e^{-3t} + c_2 e^{-3t}(-3t+1)$

$y = 0 \cdot e^{-3t} \cdot te^{-3t}$

$\boxed{y = te^{-3t}}$

$c_3 t^2 e^{-3t}$

$c_3 t^2 e^{-3t}(-3) + 2c_3 t e^{-3t}$

$c_3 t e^{-3t}(-3t+2) + c_2 e^{-3t}(-3t+1)$

$0 + c_2(1) = 1$

$0 = c_1(1) + c_2(0) \rightarrow c_1 = 0$

$1 = -3c_1(1) + c_2(1)(-3(0)+1)$

$1 = -3c_1 + c_2$

$1 = c_2$

$y(0) = 0$

$y'(0) = te^{-3t}(-3) + e^{-3t}$

$= (-3t+1)e^{-3t}$

$= (0+1)(1)$

$= 1$ ✓

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4. (15 points) Find all the real values of the constant p such that any solution to the differential equation $y'' + py' + 9y = 0$ satisfies $\lim_{t\to+\infty} y(t) = 0$. Justify your answer.

$$\frac{-p \pm \sqrt{p^2 - 4(9)}}{2}$$

if $p = 6$ or -6 solution would be $y_1 = c_1 e^{-3t}$ and $y_2 = c_2 t e^{3t}$ → does not go to 0

$$\frac{-p}{2} \pm \frac{\sqrt{36-p^2} \cdot i}{2}$$

if $-6 < p < 6$ would have $\lambda \in \mathbb{C}$.

$$c_1 e^{-p/2 t} \cos\left(\frac{\sqrt{36-p^2}}{2} t\right) + c_2 e^{-p/2 t} \sin\left(\frac{\sqrt{36-p^2}}{2} t\right)$$

$(y+1)(y+9)$

if $p > 6$ $\lambda < 0$, $\lambda_1 \neq \lambda_2$

$(y-1)(y-9)$

$$c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}$$

if $p < -6$ $\lambda > 0$

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Therefore, $\lim_{t\to\infty} y(t) = 0$ in the set of p $(-6, 6) \cup (6, \infty)$ when $p \in \mathbb{R}$

If $p = -6$, $\lambda = 3$ $e^{3t} + te^{3t}$ will go to ∞ instead of 0

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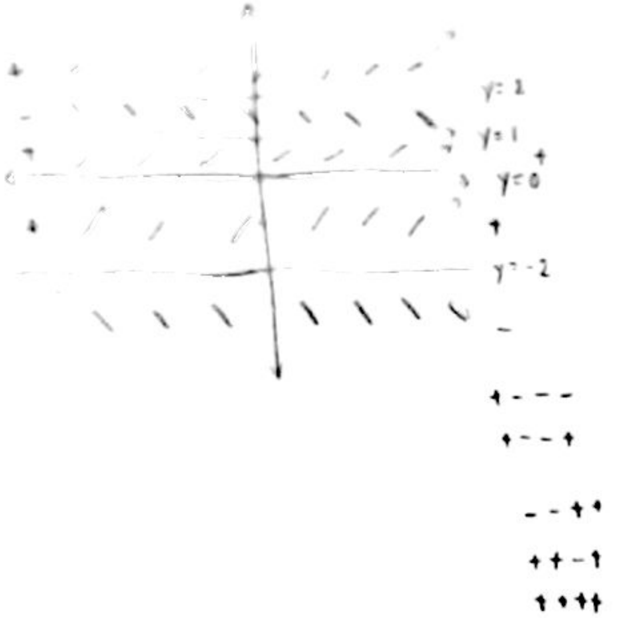
5. $dy - 3y^{\frac{1}{3}}(y-1)(y^2-4)dt = 0$

(a) (10 points) Find all the constant solutions that are (asymptotically) stable.

$\frac{dy}{dt} = 3y^{1/3}(y-1)(y-2)(y+2)$ $\frac{dy}{dt} = 0$ when $y = 0, 1, 2, -2$



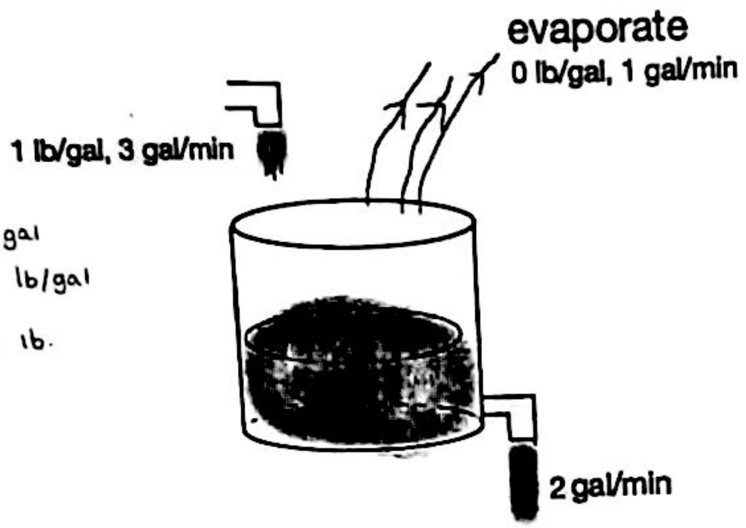
only y=1 is stable

(b) (10 points) Are the solutions to the above equation together with the initial condition $y(0) = 1$ unique? Justify your answer.

Because $\frac{dy}{dt}$ is defined everywhere, the solution is also unique everywhere. Therefore the solution to the above equation with the initial condition $y(0)=1$ is unique.

6. (20 points) The 10-gal tank is originally filled with 2-gal of salt solution with concentration 1/2 lb/gal. A spigot is opened above the tank and a salt solution with the concentration 1 lb/gal begins flowing into the tank at a rate of 3-gal/min. Meanwhile, a drain is opened at the bottom of the tank allowing the solution to leave the tank at a rate of 2 gal/min. Simultaneously, pure water is evaporating from the solution in the tank into the air at a rate of 1 gal/min. Let $x(t)$ be the amount of salt (in lbs) in the tank at the time t. Find $x(t)$.

use # in pictures.



$V_0 = 2$ gal

$C_0 = \frac{1}{2}$ lb/gal

$x_0 = 1$ lb.

let x = amount of salt

$\Delta x_{in} = 1(3) = 3$ lb/min

$\Delta x_{out} = 2\left(\frac{x}{V}\right) = x$

$\frac{dV}{dt} = +3 - 3 = 0 \rightarrow$ constant volume

$V_0 = 2$

$\Delta x = 3 - x$

$x' + x = 3$

$u = e^{\int 1 dt} = e^t$

$(e^t x)' = 3e^t$

$e^t x = 3e^t + C$

$x = 3 + Ce^{-t}$

$1 = 3 + Ce^0$

$C = -2$

$$\boxed{x = 3 - 2e^{-t}}$$