Midterm 2, Math 33b, Winter 2013
Instructor: Tonći Antunović

Printed name and student ID: ____________________

Signed name: ____________________

Section number, time and TA name: ____________________

Instructions:

- Read problems very carefully. Please raise your hand if you have questions at any time.
- The correct final answer alone is not sufficient for full credit - try to explain your answers as much as you can. This way you are minimizing the chance of losing points for not explaining details, and maximizing the chance of getting partial credit if you fail to solve the problem completely. However, try to save enough time to seriously attempt to solve all the problems.
- If it's obvious that you final answers can be simplified, please simplify them. Otherwise, your final answers need to be simplified only if this is required in the statement of the problem.
- You are allowed to use only the items necessary for writing. Notes, books, sheets, calculators, phones are not allowed, please put them away. If you need more paper please raise your hand (you can write on the back pages).

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (a) (2 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} -1, & \text{for } x \leq 1, \\ 2, & \text{for } x > 1. \end{cases}$$

Determine all possible values of x_0 for which the initial value problem

$$y' = |y+1| - f(x), \quad y(x_0) = 0$$

might have no solutions.

Solution: The right hand side $|y+1| - f(x)$ has only one discontinuity at $x_0 = 1$ so this is the only such possible value for x_0.

(b) (2 points) If y is the solution of the initial value problem

$$y' = \cos y - \sin y, \quad y(0) = 0,$$

show that $y(t) < \pi/4$ for all $t \geq 0$.

Solution: The constant function $y_1(t) = \pi/4$ is the solution of $y' = \cos y - \sin y$, $y(0) = \pi/4$. If $y(t) \geq \pi/4$ then the graphs of y and y_1 would intersect which would give two solutions for some initial value problem. This is impossible, since the right hand side $\cos y - \sin y$ is continuous in both x and y and continuously differentiable in y, any initial value problem has exactly one solution.

(c) (2 points) Let y_1 be the solution of the initial value problem

$$y' = \cos y - \sin y, \quad y(0) = 0,$$

and y_2 the solution of the initial value problem

$$y' = \cos y - \sin y, \quad y(0) = 2\pi.$$

Show that $y_2(t) = y_1(t) + 2\pi$ for all $t \geq 0$.

Solution: It is easy to check that $y = y_1 + 2\pi$ is a solution to $y' = \cos y - \sin y$, $y(0) = 2\pi$. This is because $y(0) = y_1(0) + 2\pi = 0 + 2\pi = 2\pi$ and

$$y' = y_1' = \cos y_1 - \sin y_1 = \cos(y_1 + 2\pi) - \sin(y_1 + 2\pi) = \cos y - \sin y.$$

Since there is only one solution ($\cos y - \sin y$ is continuous and differentiable) this has to agree with y_2 and so $y_2 = y_1 + 2\pi$.

(d) (2 points) Use the substitution $z = y'$ to find the general solution of the equation $y''' - y' = 0$.

Solution: Equation becomes $z'' + z = 0$ whose general solution is $z = C_1 \cos t + C_2 \sin t$ and we get the general solution y for $y''' - y' = 0$ by integrating $y = C_1 \sin t - C_2 \cos t + C_3$.

(e) (2 points) If y_1 and y_2 are two particular solutions of the differential equation $y'' - ty' + e^t y = 2\sin t$ write down a differential equation whose particular solution is $y_p = y_1 - 3y_2$.

Solution: Pluging y_p into the left hand side and using $y_1'' - ty_1' + e^t y_1 = 2\sin t$ and $y_2'' - ty_2' + e^t y_2 = 2\sin t$ gives $y_p'' - ty_p' + e^t y_p = -4\sin t$.

2. (10 points) Consider the differential equation $y' = (y^2 - 1)(y + 2)^2$. Find all equilibrium solutions and for each of them determine whether it's stable or unstable. Consider the initial value problem

$$y' = (y^2 - 1)(y + 2)^2, \quad y(0) = y_0.$$

Find all values of y_0 for which the solution of this initial value problem satisfies $\lim_{t\to\infty} y(t) = -2$. Find all values y_0 for which the solution satisfies $\lim_{t\to\infty} y(t) = -1$.

Solution: The solutions of $(y^2 - 1)(y + 2)^2 = 0$ are -2, -1 and 1. Around $y = -2$ the function $(y^2 - 1)(y + 2)^2$ is positive, at $y = -1$ it changes the values from positive to negative and at $y = 1$ from negative to positive. Drawing the phase diagram we see that $y = -1$ is stable and other two are unstable. From the direction field (or phase diagram) we see that $\lim_{t\to\infty} y(t) = -2$ holds for $y_0 \leq -2$ and $\lim_{t\to\infty} y(t) = -1$ holds for $-2 < y_0 < 1$.

3. (10 points) Find the general solution of the differential equation

$$y'' + 6y' + 9y = \frac{e^{-3t}}{t^3}.$$

Solution: The general solution to the homogeneous equation $y'' + 6y' + 9y = 0$ is $y = C_1 e^{-3t} + C_2 t e^{-3t}$. We find the particular solution using variation of parameters. We search for particular solution in the form $y_p = v_1 e^{-3t} + v_2 t e^{-3t}$. The method gives the equations

$$v_1' e^{-3t} + v_2' t e^{-3t} = 0, \qquad -3v_1' e^{-3t} + v_2'(e^{-3t} - 3te^{-3t}) = \frac{e^{-3t}}{t^3},$$

and after multiplying the equations by e^{3t}

$$v_1' + v_2' t = 0, \qquad -3v_1' + v_2'(1 - 3t) = \frac{1}{t^3},$$

We can solve by plugging $v_1' = -v_2' t$ into the second equation and we get $v_1' = -t^{-2}$ and $v_2' = t^{-3}$ and taking the antiderivatives $v_1 = t^{-1}$ and $v_2 = -\frac{1}{2}t^{-2}$ which gives

$$y_p = t^{-1} e^{-3t} - \frac{1}{2} t^{-2} t e^{-3t} = \frac{e^{-3t}}{2t}.$$

So the general solution is

$$y = C_1 e^{-3t} + C_2 t e^{-3t} + \frac{e^{-3t}}{2t}.$$

4. (10 points) Find the solution of the initial value problem

$$y'' + y = t^2 - \sin t, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution: The general solution for the homogeneous equation $y'' + y = 0$ is $y = C_1 \sin t + C_2 \cos t$.

A particular solution $y_{p,1}$ for $y'' + y = t^2$ can be searched for in the form $y_{p,1} = at^2 + bt + c$ which gives $y'_{p,1} = 2at + b$ and $y''_{p,2} = 2a$ and so

$$2a + at^2 + bt + c = t^2 \quad \Rightarrow \quad a = 1, b = 0, c = -2a = -2.$$

Therefore, $y_{p,1} = t^2 - 2$.

A particular solution $y_{p,2}$ for $y'' + y = -\sin t$ can be searched for in the form $y_{p,2} = ct \sin t + dt \cos t$ ($y_{p,2} = c \sin t + d \cos t$ will not work as this is a solution for the homogeneous equation) which gives $y'_{p,2} = (c - dt) \sin t + (ct + d) \cos t$ and $y''_{p,2} = (-ct - 2d) \sin t + (2c - dt) \cos t$ and so

$$(-ct - 2d) \sin t + (2c - dt) \cos t + ct \sin t + dt \cos t = -\sin t \quad \Rightarrow \quad c = 0, d = 1/2$$

and $y_{p,2} = \frac{t}{2} \cos t$. The general solution is then

$$y = C_1 \sin t + C_2 \cos t + t^2 - 2 + \frac{t}{2} \cos t.$$

As $y' = C_1 \cos t - C_2 \sin t + 2t + \frac{1}{2} \cos t - \frac{t}{2} \sin t$ and the initial conditions give $C_2 - 2 = 1$ and $C_1 + 1/2 = 1$ so $C_2 = 3$ and $C_1 = 1/2$ and the solution is

$$y = \frac{1}{2} \sin t + 3 \cos t + t^2 - 2 + \frac{t}{2} \cos t.$$

5. (10 points) An object of mass $m = 1$kg is attached to a spring of constant k and immersed into a liquid so that the system oscillates with the damping constant μ. The position of the object at time t is $y(t) = e^{-t} + e^{-2t}$. Find the values of k and μ as well as the initial position and velocity (at time $t = 0$). Now the system is removed from the liquid so that the damping constant is now equal to zero. The object is pulled downward from equilibrium by 1 meter and released from rest. Find the position of the object at time t and determine the amplitude and the phase of oscillations.

Solution: The initial position is $y(0) = 2$ and (since $y' = -e^{-t} - 2e^{-2t}$) the initial velocity is $y'(0) = -3$. The equation clearly has to have characteristic roots -1 and -2 and characteristic equation is $(\lambda+1)(\lambda+2) = 0$ or $\lambda^2+3\lambda+2 = 0$ so the equation looks like $y''+3y'+2y = 0$ and since the coefficient 1 by y'' agrees with the mass, the damping constant $\mu = 3$ and spring constant $k = 2$.

After taking the system out of the liquid we have $y'' + 2y = 0$ whose solution is $y = C_1 \cos(\sqrt{2}t) + C_2 \sin(\sqrt{2}t)$. As $y' = -\sqrt{2}C_1 \sin(\sqrt{2}t) + C_2\sqrt{2}\cos(\sqrt{2}t)$ the initial conditions $y(0) = -1$ and $y'(0) = 0$ give $C_1 = -1$ and $C_2 = 0$ so the solution is $y = -\cos(\sqrt{2}t)$. The amplitude is therefore $A = 1$ and the phase satisfies $\cos\phi = -1$ and $\sin\phi = 0$ so $\phi = \pi$ so we can write

$$y = \cos(\sqrt{2}t + \pi).$$