Midterm 1, Math 33b, Winter 2013
Instructor: Tonći Antunović

Printed name and student ID: ____________________

Signed name: ____________________

Section number, time and TA name: ____________________

Instructions:

- Read problems very carefully. Please raise your hand if you have questions at any time.
- The correct final answer alone is not sufficient for full credit - try to explain your answers as much as you can. This way you are minimizing the chance of losing points for not explaining details, and maximizing the chance of getting partial credit if you fail to solve the problem completely. However, try to save enough time to seriously attempt to solve all the problems.
- If it's obvious that you final answers can be simplified, please simplify them. Otherwise, your final answers need to be simplified only if this is required in the statement of the problem.
- You are allowed to use only the items necessary for writing. Notes, books, sheets, calculators, phones are not allowed, please put them away. If you need more paper please raise your hand (you can write on the back pages).

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (a) (2 points) Find all values of constant r such that the function $y(t) = (t + r)e^t$ is a solution of the equation

$$2ty' - e^{-t}y^2 = (t^2 + 2t - 4)e^t.$$

Solution:
The left hand side is

$$(2t^2 + 2tr + 2t)e^t - (t + r)^2e^t = (t^2 + 2t - r^2)e^t,$$

so we need $r^2 = 4$ and $r = \pm 2$.

(b) (2 points) General solution of the equation $x^2y' = y^2 + 2xy$ is given by $y = \frac{x^2}{C-x}$. Find the solution of the initial value problem

$$x^2y' = y^2 + 2xy, \quad y(1) = 2.$$

Solution: The initial condition gives $y(1) = \frac{1}{C-1} = 2$, so $C = 3/2$.

(c) (2 points) Is the following differential equation exact

$$(e^x y + y^2)\ dx + (e^x + x^2)\ dy = 0?$$

Solution: Differentiate $e^x y + y^2$ with respect to y, we get $e^x + 2y$. Differentiate $e^x + x^2$ with respect to x, we get $e^x + 2x$. So it's not exact.

(d) (2 points) The graph of the function y passes through the point $(1, 2)$. Moreover, for $x > 0$ the tangent line at the point $(x, y(x))$ hits the y axis at the point $(0, y(x)/2)$. Write down an initial value problem whose solution is y.

Solution: The line through $(0, y(x)/2)$ and $(x, y(x))$ has slope $\frac{y(x)}{2x}$. Therefore, the initial problem is

$$y' = \frac{y}{2x}, \quad y(1) = 2.$$

(e) (2 points) Does the following differential equation have any solutions $e^{y'} + x^2 = 0$?

Solution: The left hand side is always strictly positive, so there are no solutions.

2. (10 points) Find the solution of the initial value problem

$$y' - e^{-y} = \frac{e^{-y}}{x+1}, \qquad y(0) = 1.$$

Solution: This is a separable equation

$$y' = e^{-y}\frac{x+2}{x+1}.$$

Therefore

$$\int e^y \, dy = \int \frac{x+2}{x+1} \, dx$$

and

$$e^y = x + \ln|x+1| + C, \qquad \Rightarrow \qquad y = \ln(x + \ln|x+1| + C).$$

Clearly the equation doesn't make sense for $x = -1$ and since the initial condition is given at $x = 0$ we can assume that $x > -1$ that is $x + 1 > 0$ and so

$$y = \ln(x + \ln(x+1) + C).$$

From $y(0) = 1$ we get $C = e$ so

$$y = \ln(x + \ln(x+1) + e).$$

3. (10 points) Find the general solution of the equation

$$t^2 y' - y = e^{-1/t}.$$

Solution: This is a linear equation

$$y' - \frac{1}{t^2} y = \frac{1}{t^2} e^{-1/t},$$

and the integrating factor is

$$e^{-\int t^{-2}\ dt} = e^{1/t},$$

which gives

$$e^{1/t} y' - \frac{e^{1/t}}{t^2} y = \frac{1}{t^2},$$

or

$$\frac{d}{dt}\left(e^{1/t} y\right) = \frac{1}{t^2},$$

and

$$e^{1/t} y = \int \frac{1}{t^2} = -\frac{1}{t} + C \quad \Rightarrow \quad y = e^{-1/t}\left(-\frac{1}{t} + C\right).$$

4. (10 points) A large tank contains only 10 gallons of water with 1 pound of salt. A drain is opened which allows the mixture to leave the tank at the rate 0.5 gallons per second. At the same time pure water starts pouring in the tank at the rate 1 gallon per second raising the water level. Find the concentration of salt in the tank after time t.

Solution: If x is the mass of salt (in pounds) in the tank after time t then

$$x' = -0.5\frac{x}{10+0.5t}, \qquad \Rightarrow \qquad x(0) = 1.$$

Solving this gives

$$\int \frac{dx}{x} = -\int \frac{0.5}{10+0.5t}\, dt,$$

and

$$\ln|x| = -\ln|10+0.5t| + C \qquad \Rightarrow \qquad x = \frac{e^C}{10+0.5t}$$

(no need for absolute value since $x \geq 0$ and $10 + 0.5t > 0$). By $x(0) = 1$ we have $e^C = 10$ and so

$$x = \frac{10}{10+0.5t}.$$

Since the volume is $10 + 0.5t$ the concentration after time t is

$$\frac{10}{(10+0.5t)^2}.$$

5. (10 points) Solve the initial value problem

$$(e^x + 2xy)\ dx + (x^2 + 1)\ dy = 0, \qquad y(0) = 3.$$

Solution: This equation is exact: The partial derivative of $e^x + 2xy$ with respect to y is $2x$, the same partial derivative of $x^2 + 1$ with respect to x. Then

$$F(x, y) = \int (e^x + 2xy)\ dx = e^x + x^2 y + \phi(y),$$

which gives $\partial F/\partial y = x^2 + \phi'(y)$, and so we need $\phi'(y) = 1$ and $\phi(y) = y$. Therefore $F(x, y) = e^x + x^2 y + y$ and so the general solution to $(e^x + 2xy)\ dx + (x^2 + 1)\ dy = 0$ is

$$e^x + x^2 y + y = C, \qquad \Rightarrow \qquad y = \frac{C - e^x}{x^2 + 1}.$$

Condition $y(0) = 3$ gives $C = 4$ and so

$$y = \frac{4 - e^x}{x^2 + 1}.$$