1. (10 pts) In $\mathbb{R}^3$, find the projection of a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ on to the plane $x - 2y + 6z = 0$.

projection of $\vec{v}$ onto plne $x-2y+6z=0$

do an orthogonal projection

$$[1 \quad -2 \quad 6 \;\vdots\; 0]$$

$x_1 = 2s - 6t$
$y = s$
$z = t$

ker(plne) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix} \right\}$

find an orthornonmal basis for

$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$v_2^\perp = v_2 - (u_1 \cdot v_2) u_1$

$= \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right.$

$\rightarrow \frac{1}{5}(-12) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\frac{1}{5} \begin{bmatrix} -24 \\ -12 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix} - \begin{pmatrix} \frac{-24}{5} \\ \frac{-12}{5} \\ 0 \end{pmatrix}$

$\begin{bmatrix} \frac{-30}{5} + \frac{24}{5} \\ 0 + \frac{12}{5} \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} -\frac{6}{5} \\ 12/5 \\ 1 \end{bmatrix} = v_2^\perp$

$v_2 = \frac{v_2^\perp}{\|v_2^\perp\|} = \quad \|v_2^\perp\| =$

$\frac{36}{25} + \frac{144}{25} + \frac{25}{25}$

$\frac{205}{25}$

$v_2 = \frac{5}{\sqrt{205}} \begin{bmatrix} -6/5 \\ 12/5 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-6}{\sqrt{205}} \\ \frac{12}{\sqrt{205}} \\ \frac{5}{\sqrt{205}} \end{bmatrix}$

cont'd → on back

$$\text{Ortho normal basis: } \left\{ \begin{array}{c} u_1 \\ \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} u_2 \\ \frac{1}{\sqrt{205}}\begin{bmatrix} -6 \\ 12 \\ 5 \end{bmatrix} \end{array} \right\}$$

given that we have an orthonormal basis

we can find the projection of a vector V onto

the plane $x-2y+6z=0$ with the

equation $V - (u_1 \cdot V)u_1 - (u_2 \cdot V)u_2$

since $(u_1 \cdot v)(u_1) + (u_2 \cdot v)u_2$ is the orthogonal

projection

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2. (10 pts) Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.
Let T be the reflection about $y = x$. What is the matrix for the linear transformation T under the basis $\mathcal{B}$?

$S = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\quad A =$ reflection about $y = x$ $\quad$ find reflection first

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B = S^{-1} A S$$

$$S^{-1} = \frac{1}{4-6} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$B = -\frac{1}{2} \underbrace{\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\downarrow$$

$$B = \frac{1}{2} \begin{bmatrix} -3 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$- \frac{1}{2} \begin{bmatrix} 5 & 7 \\ -3 & -5 \end{bmatrix}$$

$$\boxed{B = - \begin{bmatrix} 5/2 & 7/2 \\ -3/2 & -5/2 \end{bmatrix}}$$

10

3. (10 pts) Consider vectors $\vec{v}_1 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, and $\vec{v}_2 = \begin{bmatrix}2\\4\\2\\8\end{bmatrix}$ in $\mathbb{R}^4$. Assume $V = \text{span}\{\vec{v}_1, \vec{v}_2\}$. Find an orthonormal basis for $V^{\perp}$.

$V^{\perp} = (\operatorname{im} V)^{\perp} = \ker(V^T)$

$V^T \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 2 & 8 \end{bmatrix}$

$2(\mathrm{I}) - (\mathrm{II})$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 4 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\quad \mathrm{I} - 3(\mathrm{II})$

$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$x_2 = s \quad x_4 = t$

$x_1 = -2s - 4t$

$x_2 = s$

$x_3 = 0$

$x_4 = t$

$s\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-4\\0\\0\\1\end{bmatrix}$

find orthonormal basis given a basis $\left\{ \begin{bmatrix}-2\\1\\0\\0\end{bmatrix} \begin{bmatrix}-4\\0\\0\\1\end{bmatrix} \right\}$

$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$

$v_2^{\perp} = v_2 - (u_1 \cdot v_2)u_1$

$v_2^{\perp} = \begin{bmatrix}-4\\0\\0\\1\end{bmatrix} - \left(\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}-4\\0\\0\\1\end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$

$v_2^{\perp} = \begin{bmatrix}-4\\0\\0\\1\end{bmatrix} - \frac{1}{5}(8)\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$ $\qquad -\frac{20}{5} + \frac{16}{5}$

$v_2^{\perp} = \begin{bmatrix}-4\\0\\0\\1\end{bmatrix} - \begin{bmatrix}-16/5\\8/5\\0\\0\end{bmatrix}$

$v_2^{\perp} = \begin{bmatrix}-4/5\\-8/5\\0\\1\end{bmatrix}$

$u_2 = \frac{v_2^{+}}{\|v_2^{+}\|} = \overset{\|v_2^{+}\|}{\sqrt{\frac{16}{25} + \frac{64}{25} + \frac{25}{25}}}$

$\frac{5}{\sqrt{105}}\begin{bmatrix}-4/5\\-8/5\\9\end{bmatrix}$ $\quad \frac{105}{25} = \quad u_2 \frac{1}{\sqrt{105}}\begin{bmatrix}-4\\-8\\0\\5\end{bmatrix} \rightarrow$

$v_1 \quad \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$v_2 = \frac{1}{\sqrt{105}} \begin{bmatrix} -4 \\ -8 \\ 0 \\ 5 \end{bmatrix}$

$64 + 16$
$+25$

$$\boxed{\text{orthonormal basis of } V^{\perp}: \left\{ \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4/\sqrt{105} \\ -8/\sqrt{105} \\ 0 \\ 5 \end{bmatrix} \right\}}$$

4. (10 pts) Compute the determinant of the matrix

$$\begin{pmatrix} 4 & 10 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 5 & 10 & 2 \\ 8 & 20 & 33 & 15 \end{pmatrix}.$$

Using block matrices we see that we can get a det of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

where $\det\begin{pmatrix} 4 & 10 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 5 & 10 & 2 \\ 8 & 20 & 33 & 15 \end{pmatrix}$

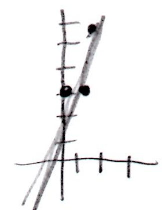
$= \det\begin{bmatrix} 4 & 10 \\ 1 & 2 \end{bmatrix} \cdot \det\begin{bmatrix} 10 & 2 \\ 33 & 15 \end{bmatrix}$

$8-10$
$=-2$

$150-66$
$=84$

$\boxed{\text{det of matrix: } -168}$

10



5. (10 pts) Fit a linear function of the form $f(t) = c_0 + c_1 t$ to the data points $(0, 3), (1, 3), (1, 6)$ using Least-Squares.

$A\vec{x} = \vec{b}$

find A

$c_0 = 3$

$c_1 + c_0 = 3$

$c_1 + c_0 = 6$

$Ax + b =$

$Ax^2 + by + z = 0$

$$\begin{array}{cc} c_1 & c_0 \end{array}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 3 \\ a \end{bmatrix}$$

$$A^T A \vec{x} = A^T b$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$$

2×3

$6 - 4$

$$\vec{x} = \underbrace{\frac{1}{2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$$

$$\frac{1}{2} \begin{bmatrix} -2 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$$

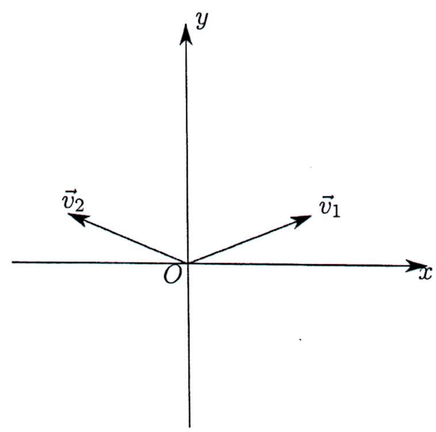
$-6 + 3 + 6$

$$= \frac{1}{2} \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\boxed{f(t) = \frac{3}{2} t + 3}$$

10

6. (bonus 10 pts) The two vectors $\vec{v}_1 = \begin{bmatrix} \cos(\pi/6) \\ \sin(\pi/6) \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -\cos(\pi/6) \\ \sin(\pi/6) \end{bmatrix}$ form a basis for $\mathbb{R}^2$. See Figure 1 for an illustration. Denote the basis by $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$. Assume the vector $\vec{u_1} \in \mathbb{R}^2$ has coordinates $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ under the basis $\mathcal{B}$, formally written as $[\vec{u_1}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Rotate $\vec{u}_1$ counter-clockwise by $2\pi/3$ and obtain another vector $\vec{u}_2$. What is $[\vec{u}_2]_{\mathcal{B}}$?



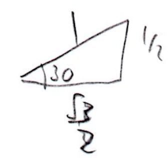


Figure 1: Illustration

rotate counter clockwise

Find $\vec{v_1}$ in standard basis then rotate

$\vec{x} = S\,[\vec{x}]_B$

$$\begin{bmatrix} \sqrt{3}/2 & -\sqrt{3}/2 \\ 1/2 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{2\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} \qquad \vec{x} = \begin{bmatrix} -\sqrt{3}/2 \\ 5/2 \end{bmatrix}$$

$\frac{5}{2}$

rotate counter clockwise $\begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix}$

$$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{3}}{2} \\ 5/2 \end{bmatrix}$$

$\frac{\sqrt{3}}{4} - \frac{5\sqrt{3}}{4} = -\frac{4\sqrt{3}}{4} =$ $\begin{bmatrix} -\sqrt{3} \\ -2 \end{bmatrix}$ $\quad \frac{-3}{4} - 5/4 = -8/4 = -2$

for v_2 in standard basis

now translate back into $[v_2]_B$

10

$$S^{-1}\vec{x} = [\vec{x}]_B$$

$$\frac{2}{\sqrt{3}}\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} -\sqrt{3} \\ -2 \end{bmatrix}$$

$$\frac{2}{\sqrt{3}}\begin{bmatrix} \frac{-\sqrt{3}}{2} - \frac{2\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2} \end{bmatrix}$$

$$\frac{2}{\sqrt{3}}\begin{bmatrix} \frac{-3\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} \end{bmatrix} = \boxed{[v_2]_B = \begin{bmatrix} -3 \\ -1 \end{bmatrix}}$$

$$S = \begin{bmatrix} \sqrt{3}/2 & -\sqrt{3}/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$S^{-1} = \frac{1}{\frac{\sqrt{3}}{4} + \sqrt{3}/4}\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$= \frac{2}{\sqrt{3}}\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$